CHAPTER 5
MATRICES

NOTES

Definition of matrix:

A matrix of order $m \times n$ is a rectangular array of mn numbers, arranged in m rows (horizontal) and n columns (vertical).

Notes: 1) A matrix is usually represented by capital letter. For instance a matrix with m rows and n columns may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

2) The suffixes i and j in the element a_{ij} indicate the number of row and column in which the element occurs. The above matrix A is also represented by the symbol $A = \left[a_{ij} \right]$.

ORDER OF A MATRIX:

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix.

Diagonal elements:

An element a_{ij} of a matrix $A = \left[a_{ij} \right]$ is called the diagonal element if $i = j$.

TYPES OF MATRICES:

1. **Rectangular matrix:** Any matrix of order $m \times n$ (where m is not necessarily equal to n) is called a rectangular matrix.

   For example: $\begin{bmatrix} 2 & 3 \\ -5 & 5 \\ 7 & 1 \end{bmatrix}$ is a rectangular matrix.

2. **Square matrix:** A matrix in which the number of rows is equal to the number of columns is called a square matrix.

   Thus, any $n \times n$ matrix is known as a square matrix of order n

   or n-rowed square matrix.

   For example: $A = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 4 & 6 \\ 0 & 9 & 7 \end{bmatrix}$ is a square matrix of order 3

3. **Row matrix:** A matrix having only one row is called row matrix.

4. **Column matrix:** A matrix having only one column is called a column matrix.

For example: $B=\begin{bmatrix}3\\7\\1\end{bmatrix}$

5. **Diagonal matrix:** A square matrix $A=\left[a_{ij}\right]$ is said to be diagonal matrix if all its non-diagonal elements are zero. Thus $A=\left[a_{ij}\right]_{n\times n}$ is a diagonal matrix, if $a_{ij}=0$ for $i\neq j$.

For example: $A=\begin{bmatrix}5&0&0\\0&6&0\\0&0&7\end{bmatrix}$

6. **Scalar matrix:** A diagonal matrix whose diagonal elements are all equal , is called a scalar matrix.

For example: $A=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$

7. **Identity matrix (or Unit matrix):** A square matrix in which the diagonal elements are all 1 and the rest are all zero is called an identity matrix.

For example: $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

8. **Zero matrix:** A matrix is said to be zero or null matrix if all the elements are zero.

For example: $\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

9. ## Triangular matrices:

(i) Upper triangular matrix: A square matrix all of whose elements below the principal diagonal are zero, is called an upper triangular matrix.

$$\textbf{For example: } \begin{bmatrix} 4 & 7 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) Lower triangular matrix: A square matrix all of whose elements above the principal diagonal are zero, is called an lower triangular matrix.

$$\textbf{For example: } \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 9 & 6 \end{bmatrix}$$

EQUALITY OF MATRICES:

Definition: Two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are said to be equal if

(i) they are of the same order.

(ii) each element of *A* is equal to the corresponding element of *B*, that is

$a_{ij} = b_{ij}$ for all admissible values of *i* and *j*.

OPERATION ON MATRICES:

(a) Addition of matrices:

Let *A* and *B* the two matrices of the same order. Then the sum of *A* and *B*, denoted by *A*+*B* is defined as the matrix, each elements of which is the sum of the corresponding elements of *A* and *B*.

Example if

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\text{Then, } A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+a_{23} \end{bmatrix}$$

Note: Two matrices *A* and *B* are conformable for addition if *A* and *B* have the same numbers of rows and columns.

(b) Multiplication of a matrix by a scalar:

If k is a scalar and A is a matrix, then the product kA is defined as the matrix obtained on multiplying each element of A by k.

For example, if $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, then for any scalar k

$$kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}$$

(c) Subtraction of Matrices:

If A and B are two matrices of the same order, then the difference $A - B$ is defined as a matrix obtained by subtracting the corresponding element of B from the corresponding elements of A.

Example $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

Then, $A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \end{bmatrix}$

Notes: **1)** Matrix addition is commutative as well as associative

i.e. (i) A + B = B + A

(ii) (A+B)+C = A+(B+C) ,whenever A, B, C are matrices of same order.

2) The distributive law $k(A+B) = kA + kB$ hold for any scalar k and A and B are two matrices of same order.

(d) Multiplication of matrices

Let A and B be the two matrices such that the number of columns of A is equal to the number of rows of B. Then the two matrices A and B are said to be conformable for the product AB. And the product AB is defined only when A and B are conformable for this product.

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Then, the product *AB* is defined as

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + b_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

Here, we observed that *A* is 3×3 matrix and *B*, 3×2 matrix, then the product *AB* is 3×2 matrix.

Notes: **1)** If *A* is $m \times n$ matrix and *B*, $n \times p$ matrix, then the product *AB* is an $m \times p$ matrix.

2) Matrix multiplication is not commutative.

i.e. $AB \neq BA$

3) Matrix multiplication is associative:

i.e. $(AB)C = A(BC)$; provided A, B, C are conformable for their corresponding products.

4) Matrix multiplication holds distributive law:

$A(B+C) = AB + AC$; provided A, B, C are conformable for the products and sum.

TRANSPOSE OF A MATRIX:

Given a matrix *A*, then the matrix obtained from *A* by changing its rows into columns and columns into rows is called the transpose of *A*. Transpose of *A* is denoted by A' or A^t

For example for a given matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Transpose of *A* i.e. $A' = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix}$

Notes: **1)** If *A* is $m \times n$ matrix then A' will be $n \times m$ matrix.

2) The element in the i^th^ row – j^th^ column of *A* i.e. $(i, j)^{th}$ element of *A* ,becomes the $(j,i)^{th}$ element of A'.

Theorems (Properties of transpose of Matrix):

If A', B' denote the transpose of A and B respectively, then

i) $(A')' = A$

ii) $(A+B)' = A' + B'$, A, B being conformable for addition.

iii) $(kA)' = kA'$ where k is any scalar.

iv) $(AB)' = B'A'$ A, B being conformable for multiplication.

Symmetric Matrix:

A square matrix A is said to be symmetric if $A' = A$ i.e. if $a_{ij} = a_{ji}$

For example, $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -15 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric as $A' = A$.

Skew-Symmetric Matrix:

A square matrix A is said to be Skew-symmetric if $A' = -A$ i.e. if $a_{ij} = -a_{ji}$.

For example, $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$ is a skew-symmetric as $A' = -A$.

Note: Every diagonal element of a skew-symmetric matrix is necessarily zero.

Theorem: Every square matrix can be expressed in one and only one way, as a sum of a symmetric matrix and a skew-symmetric matrix.

Note: If A be is any square matrix, then we can write

$$A = \left\{\frac{1}{2}\left(A + A'\right)\right\} + \left\{\frac{1}{2}\left(A - A'\right)\right\}.$$

= {symmetric} + {skew-symmetric}

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