Chapter–1

SET, RELATIONS AND FUNCTIONS

1.1. Introduction :

In this chapter, we shall introduce the concept of set and some other related terms which are of fundamental importance in all branches of mathematics. Set theory was first developed by George Cantor (1845–1918), the great German mathematician. Since then it has become so important that almost all the mathematical theories propounded thereafter make frequent use of set theoritical concepts. Modern pure mathematics can be regarded as the study of sets relative to assigned structures.

1.2. Set :

In certain branches of mathematics, we have to make use of certain undefined terms. For example, in Euclidean geometry properties of points, lines and planes are discussed although these are undefined terms. In set theory also set itself is an undefined term. By defining a set as a collection (of objects) we are simply asserting that a set is a set. To impart to you a little idea of the undefined term 'set' we use the following so called definition.

Definition : A well-defined collection of distinct objects is called a set.

When we say a well-defined collection, it is meant that there exists a rule (or rules) by which it can be determined whether a given object does or does not belong to the collection. For example, a collection of "any five natural numbers" is not a set, because the objects in this collection are not well-defined. Given a natural number say 3, it may or may not belong to the collection. However, the collection of "the first five natural numbers" is a set. Its objects are well-defined and they are 1, 2, 3, 4, and 5.

The objects in a set are called *elements or members* of the set. The elements in a set are different from each other i.e. no element is repeated. Also, given any object, it must be either in the set or not in the set but not both. These characteristics are described by saying that the elements in a set are *distinct and distinguishable.*

Consider the collection X of all persons shaved by a barber who lives in a certain village and who shaves those and only those who live in the village and do not shave themselves. Although this collection seems to be well-defined, it is not a set, for one cannot tell whether the barber himself is a member of this collection or not. Suppose the barber is a member of X. Then he shaves himself and he cannot be a member of X as X contains only those who do not shave themselves. Again, suppose the barber is not a member of X. Then he does not shave himself and therefore, the barber as a man shaving those who do not shave themselves, will shave himself and hence he must be a member of X. It is seen that the barber is a member of X and not a member of X at the same time.

Thus, we arrive at a paradox known as *Russel Paradox* after the British logician and philosopher *Bertrand Russel* (1872–1970), who first drew attention to this paradox.

In the following, some important sets of numbers which occur frequently in our discussion, are given with their usual notations :

N = the set of natural numbers.

Z(or I) = the set of integers.

Q = the set of rational numbers.

Q^+ = the set of positive rational numbers.

R = the set of real numbers.

R^+ = the set of positive real numbers.

R^- = the set of negative real numbers.

We shall denote sets by capital letters A, B X, Y etc. and elements in a set or sets by small letters *a, b, x, y* etc. We write $x \in A$ (read as '*x* belongs to A') to mean that the element *x* belongs to the set A and $x \notin A$ (read as '*x* does not belong to A') to mean that *x* is not a member of A.

For example, $3 \in N$ (for 3 is a natural number) and $-3 \notin N$ (for –3 is not a natural number.)

Some symbols which are used very often in set theory are given below along with their meaning :

(i) : such that

(ii) $\forall$ for every (for all)

(iii) $\exists$ there exist(s)

(iv) $\Rightarrow$ implies (imply)

(v) $\Leftrightarrow$ implies and is implied by (imply and are implied by)

(vi) *iff* if and only if.

1.3. Representation of a set :

There are two forms in which a set may be represented (a) *tabular or roster form* and (b) *set builder (or rule) form*.

(a) A set is specified in the tabular form by listing all its elements within a pair curly (second) brackets. Thus the set of the first five natural numbers is represented by {1, 2, 3, 4, 5} which we refer to as the *Tabular or Roster Form* of the set. It may be noted that the elements within the brackets are separated by commas. This form is convenient and can be used without ambiguity when the set has only a few elements.

(b) A set may also be specified by stating a property which is satisfied by each of its elements and not by any other element. Thus, the set of all elements x having the property $P(x)$ is represented by $\{x : P(x)\}$ which is then called the *Set-builder Form of the set*. In this form, the set of the first five natural numbers can be written as

$\{x : x$ is a natural number less than $6\}$ or $\{x : x \in N, x<6\}$.

Example 1. Exhibit in the tabular form the following sets :

(i) set of vowels in the English alphabet.

(ii) set of integers greater than 5 but less than 20.

(iii) set of letters of the word 'algebra'.

Solution : (i) $\{a, e, i, o, u\}$

(ii) $\{6, 7, 8, \ldots\ldots\ldots., 18, 19\}$

(iii) $\{a, l, g, e, b, r\}$

Example 2. Rewrite the set in the above example in the set builder form.

Solution : (i) $\{x : x$ is a vowel of the English alphabet$\}$

(ii) $\{n : n \in Z, 5<n<20\}$

(iii) $\{x : x$ is a letter of the word 'algebra'$\}$

1.4. Null set, Singleton set, Finite set and Infinite set :

A set having no element is called an *empty set* or a *void set* or a *null set* and it is denoted by the Greek letter ϕ (phi).

Each of the following sets is an empty set :

(i) The set of integers greater than 5 and less than 2.

(ii) The set of rational numbers whose square is 2.

(iii) $\{x : x \neq x\}$

A set having at least one element is called a *non-empty set*.

A set having exactly one element is called a *singleton set*. Thus, $\{a\}$ is a singleton set consisting of the single element 'a'. The set of integers greater than 1 and less than 3, is also a singleton set consisting of only integer 2.

A set containing only a finite number of elements is called a *finite set*.

Each of the sets ϕ, $\{a\}$, $\{1, 2, 3, 4\}$ is a finite set. The set of all positive integers less than 100 is also a finite set having only ninetynine elements.

A set containing an infinite number of elements is called an *infinite set*.

Each of the sets N, Z, Q, R is an infinite set. The set of even integers is also an infinite set.

1.5. Subsets and Supersets :

If A and B are sets such that all elements in B are also in A, then B is said to be a *subset* of A and A is said to be a *superset* of B.

Symbolically, we write $B \subseteq A$(read as 'B is contained in A') or $A \supseteq B$ (read as 'A contains B') to mean that 'B is a subset of A' or equivalently that 'A is a superset of B'. From definition, it follows that $B \subseteq A$ *iff* $x \in B \Rightarrow x \in A$.

If A = {1, 2, 3, 4, 5} and B = {2, 3, 5} then $B \subseteq A$ as each element in B is also in A i.e. $x \in B \Rightarrow x \in A$. And if A is the set of even integers and B = {1, 2, 3, 4} then B is not a subset of A for $1 \in B$ but $1 \notin A$.

The empty set ϕ is a subset of any set A, because every element in ϕ is also in A (as there is no element in ϕ). Also any set A is a subset of itself (why ?). Thus every no-empty set has at least two subsets, namely ϕ and the set itself.

1.6. Equality of sets :

Two sets A and B are said to be *equal* if each is a subset of the other i.e. if each element in A is also in B and each element in B is also in A.

We write A = B to mean, as usual that A and B are equal and $A \neq B$, to mean that A and B are unequal. By definition, A = B *iff* $x \in A \Leftrightarrow x \in B$.

The sets {1, 2, 3} and {3, 1, 2} are equal for each element in one is also in the other. The sets {1, 5} and $\{x : x^2 - 6x + 5 = 0\}$ are also equal (why ?). The sets A={1, 2, 3} and B={2, 3, 4} are unequal, for $1 \in A$ but $1 \notin B$.

1.7. Proper and Improper subsets :

A set B is said to be a *proper subset* of a set A if B is a subset of A and B is not equal to A.

Thus, B is a proper subset of A if and only if all elements in B are in A and there exists at least one element of A which is not in B.

We shall write $B \subset A$ to mean that B is a proper subset of A.

A subset of a set is said to be *improper* if it is not proper.

It is clear that for any given non-empty set A, all subsets other than A are proper and the only improper subset is A itself. (However some authors treat the empty set ϕ as an improper subset of any non-empty set.)

If A={1, 2, 3, 4, 5} and B={1, 2, 4, 5}, then B is a proper subset of A, because every element in B is also in A and there is one element, namely 3 in A which is not in B.

Example : Write down all subsets of the set $\{a, b, c\}$. Pick out proper subsets.

Solution : The subsets of the set $\{a, b, c\}$ are as listed below :

ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b, c\}$.

All subsets except the last one are proper subsets.

1.8. Power set :

Let A be a set. Then the set containing, as its elements, all the subsets of A is called the *power set* of the set A and denoted by the symbol P(A) or 2^A.

Symbolically, $P(A)=\{B : B\subseteq A\}$

Evidently, ϕ and A are both elements of P(A).

If $A=\{1, 2\}$, then its subsets are ϕ, $\{1\}$, $\{2\}$ and $\{1,2\}$. Hence its power set i.e. P(A) is the set $\{\phi, \{1\}, \{2\}, A\}$. Observe that although A itself is an element of P(A), the elements 1 and 2 which are in A, are not elements of P(A). The elements of P(A) are sets.

1.9. Set of sets :

A set whose elements are sets themselves, is known as a *set of sets*.

For example, if A is a set then P(A) is a set of sets. The set $\{\phi\}$ is a set of sets for it has the empty set as its only element. The set $\{1, 2, \{3, 4\}\}$ is not a set of sets, for although it has the set $\{3, 4\}$ as one of its elements, there are other elements viz 1 and 2 which are not sets themselves. But $\{\{1\}, \{2\}, \{3, 4\}\}$ is a set of sets.

1.10. Equivalent Sets :

If A is a finite set, then we denote by $n(A)$ the number of elements in A. For example, if A is the set of even positive integers less than 9, then $n(A)=4$ (why ?) and if $B=\{a, b, c\}$, then $n(B)=3$.

Two finite sets A and B are said to be *equivalent* if $n(A)=n(B)$.

We write $A\cong B$ or $A\sim B$ to mean that A and B are equivalent.

Thus, if $A=\{1, 2, 3\}$ and $B=\{a, b, c\}$, then $A\sim B$, for $n(A)=3=n(B)$.

Example : Which of the following statements are true ? Give reasons.

(i) $\{a\}\in\{a, b, c\}$

(ii) $b\in\{a, b, c\}$

(iii) $\phi\in\{a, b, c\}$

(iv) $\phi\subseteq\{a, b, c\}$

(v) $\phi\subseteq\{\phi\}$

(vi) $\phi\in\{\phi\}$

(vii) $\{a, b, c\}=\{c, a, b\}$

Solution : (i) False, for $\{a\}$ is a set itself and not an element of $\{a, b, c\}$ whose elements are the letters a, b, and c.

(ii) True, for b is an element of the set $\{a, b, c\}$.

(iii) False, for the empty set ϕ is not an element of the set.

(iv) True, for ϕ is a subset of any set.

(v) True, for ϕ is a subset of any set.

(vi) True, for ϕ is an element of the set $\{\phi\}$

(vii) True, for each element in one of the sets is also an element of the other.

EXERCISE 1.1

1. Define a set. Give an example. Is there any collection which is not a set ? If so, give an example.
2. What are the different ways to specify a set ? Explain with examples.
3. Rewrite the following sets in the tabular form :

   (i) $\{x : x$ is factor of $20\}$ (ii) $\{x : x$ is a letter of the word 'collection'$\}$

   (iii) $\{x : x$ is a digit in the number $552327\}$

   (iv) $\{x : x$ is a prime number lying between 4 and $20\}$
4. Represent the following sets in set builder form :

   (i) $\{2, 5, 6, 9\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7\}$ (iii) $\{2, 4, 6, 8\}$

   (iv) $\{3, 6, 9, 12 ....\}$ (v) $\{f, a, c, t, o, r\}$
5. State whether the following statements are true or false :

   (i) $\phi = 0$ (ii) $0 \in \phi$ (iii) $0 \subseteq \phi$ (iv) $\phi \subseteq \phi$

   (v) $\phi \in \phi$ (vi) $\{a\} \subseteq \{a, b, c\}$ (vii) $a \subseteq \{a, b, c\}$ (viii) $a \in \{a, b, c\}$

   (ix) $\{b\} \in \{a, b, c\}$ (x) $\{1, 2\} = \{2, 1\}$ (xi) $\{1, 2, 3\} \neq \{2, 1, 3\}$

   (xii) $\{2, 5, 6\} = \{x : x$ is a digit in $6525\}$ (xiii) $A \subseteq P(A)$ (xiv) $A \in P(A)$

   (xv) $\{A\} \subseteq P(A)$ (xvi) $\{A\} \in P(A)$

   (xvii) Every subset of a finite set is finite.

   (xviii) Every subset of an infinite set is in

   (xix) Every set has a proper subset.

   (xx) Every set has at least two distinct subsets.

   (xxi) A non-empty set has at least two subsets.

6. Which of the following sets are finite ?
   (i) The set of days in a week.
   (ii) The set of insects living on earth.
   (iii) The set of integers less than 10.
   (iv) The set of natural numbers greater than 10.
   (v) The set of prime numbers.
   (vi) The set of positive integers less than 100.
   (vii) The set of even primes.
   (viii) The set of rational numbers lying between 1 and 2.
7. Which of the following pairs of sets are equal ?
   (i) $\{x : x$ is a letter in 'enter'$\}$ and $\{x : x$ is a letter in 'rent'$\}$
   (ii) $\{x : x$ is a positive integer$<4\}$ and $\{x : x$ is a digit in 331212$\}$
   (iii) $\{x : x \geq 0$ and $x \leq 0\}$ and ϕ (iv) ϕ and $\{\phi\}$.
8. When are two finite sets said to be equivalent ? Will all equivalent sets be equal ? Justify your answer with the help of an example.
9. Which of the following sets are empty ?
   (i) $\{x : x \neq x\}$ (ii) $\{x : x$ is a real number whose square is not positive$\}$
   (iii) $\{x : x \in \mathrm{R}$ and $x^2 < 0\}$ (iv) $\{x : x \in \mathrm{N}$ and $x^2 = 0\}$ (v) $\{x : x \in \mathrm{R}$ and $x^2 = 0\}$
   (vi) $\{x : x$ is a prime number divisible by 3$\}$
   (vii) $\{x : x$ is a prime number divisible by 6$\}$
   (viii) $\{x : x$ is an odd integer divisible by 2$\}$

1.11. Universal set :

In set theory, all sets under a discussion are taken as subsets of a fixed set. We call this fixed set a *universal set or universe of discourse* and it is usually denoted by the letter U. Thus all elements, in a discussion, belong to the universal set for the discussion.

Example : 1. In the study of plane geometry, we deal with plane figures such as straight lines, triangles, circles, ellipses, parabolas, etc. all of which are sets of points belonging to a plane. Here, the plane (considered as a set of points) is the universal set, for all the sets under discussion are its subsets.

2. In the study of properties of real numbers, R is the universal set.

1.12. Venn diagrams :

Diagrams are often used to illustrate the relations which may exist between sets. These diagrams are called *Venn-Euler diagrams* or simply *Venn-diagrams*. In these diagrams, a universal set is represented by the points in a rectangle and its subsets by points within circles (or some other simple closed curves) drawn inside the rectangle.

Example : To illustrate that a set B is a subset of a set A, the following Venn-diagram can be drawn.

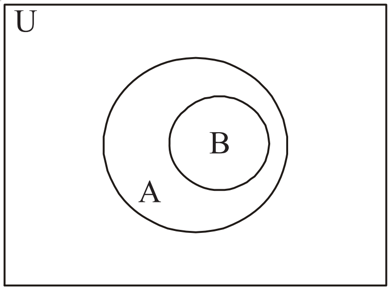


Fig. 1.1

1.13. Basic operations on sets :

We are familiar with the operations of addition, subtraction etc. in the set of numbers. By adding two integers we get an integer and we say that addition is an operation in the set of integers. Let us observe the fact that addition gives unique integer corresponding to a given pair of integers. It can be looked upon as a rule which assigns to each (ordered) pair (m, n) of integers a unique i.e. one and only one integer namely $m+n$. If there is a rule which associates each ordered pair of sets with a unique set, then we shall call the rule *a set operation* or *an operation on sets*. Thus, set operations are rules of forming a set corresponding to a given ordered pair of sets. Here, we shall discuss about some basic set operations.

1.14. Union :

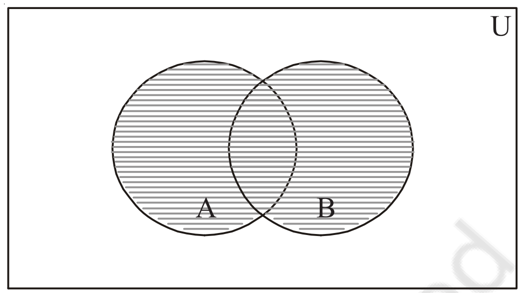
Consider the sets A={1,2,3,4} and B={2,4,6,8}. Let us examine which elements belong to A or B or both. The elements 1 and 3 belong to A alone ; 2 and 4 belong to both A and B, and 6 and 8 belong to B alone. Collecting all these elements which belong to A or to B or to both A and B, we get the set {1,2,3,4,6,8}. This set is called the union of the sets A and B under discussion.

Definition : The *union* of two sets A and B denoted by A∪B, (read as 'A union B' or 'A cup B') is defined as the set of all elements which belong to at least one of A and B.

Thus $A \cup B = \{x : x \in A \text{ or } x \in B\}$ where the word 'or' is used inclusively i.e. when we write '$x\in$ A or $x\in$ B' we mean that x is in A alone or in B alone or in both A and B .

1.15. Venn diagram for union :

If A and B are the sets represented by the respective circular regions then $A \cup B$ is the entire shaded region.



$A \cup B$

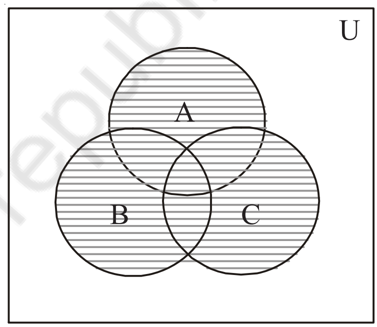
Fig. 1.2

1.16. Union is associative :

If A, B, C are any sets, then

$(A \cup B) \cup C = A \cup (B \cup C)$ .... (1)

for each of them is the set of elements which belong to at least one of A, B and C (Refer Fig 1.3). Each of these will be denoted by $A \cup B \cup C$.



$A \cup B \cup C$

Fig. 1.3

The property of set union defined by the relation (1) is expressed by saying that set union is an associative operation.

1.17. Union is commutative :

For any two sets A and B,

$A \cup B = B \cup A$ .... (2)

The proof of (2) is quite obvious for,

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\}$

$= B \cup A$ (by definition)

Observe that the statements "$x \in A$ or $x \in B$" and "$x \in B$ or $x \in A$" have the same meaning.

The relation (2) is expressed by saying that set union is commutative.

Example 1 : For any set A, prove that

$A \cup A = A$ .... (3)

[The relation (3) is described by saying that union satisfies *idempotent law*]

Proof : $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

2. Prove that (i) $A \subseteq A \cup B$ (ii) $B \subseteq A \cup B$

Proof : (i) If $x \in A$, then by definition of union, $x \in A \cup B$. In other words every element in A is also in $A \cup B$. $\therefore A \subseteq A \cup B$

(ii) Similarly, $B \subseteq A \cup B$

3. For any set A, $A \cup \phi = A$ [i.e. ϕ is identity for set union]

Proof : Clearly $A \subseteq A \cup \phi$ .... (i) (why ?)

If x is an element of $A \cup \phi$, then clearly $x \in A$ or $x \in \phi$. But there is no element x in ϕ. It follows that $x \in A$. Thus every element in $A \cup \phi$ is in A

i.e. $x \in A \cup \phi \Rightarrow x \in A$. Therefore $A \cup \phi$ must be a subset of A.

i.e. $A \cup \phi \subseteq A$ .... (ii)

From (i) and (ii), by definition of equality of sets, it follows that,

$A \cup \phi = A$.

1.18. Union of finite number of sets :

The union of n sets ($n \in N$), $A_1, A_2, A_3, \ldots\ldots, A_n$ denoted by $A_1 \cup A_2 \cup \ldots\ldots \cup A_n$ or by $\bigcup_{i=1}^{n} A_i$ is the set of elements which belong to at least one of the n sets. Thus,

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots\ldots \text{ or } x \in A_n\}$$

$$= \{x : x \in A_i \text{ for at least one } i \in \{1, 2, \ldots\ldots, n\}\}$$

Example : If $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$, $C = \{1, 3, 5\}$, $D = \{5, 7, 9\}$ and $E = \{3, 5, 7, 9\}$ then the elements belonging to at least one of these 5 sets are 1,2,3,4,5,6,7 and 9.

Hence $A \cup B \cup C \cup D \cup E = \{1, 2, 3, 4, 5, 6, 7, 9\}$

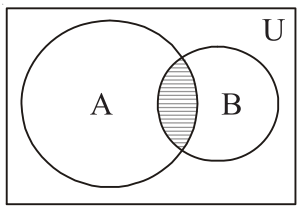
1.19. Intersection :

Consider the sets $A = \{a, b, c, d, e\}$ and $B = \{b, d, p, q\}$. What are the elements which belong to both A and B ? They are b and d. The set $\{b, d\}$ is called *intersection* of the two sets A and B.

Definition : The *intersection* of two sets A and B, denoted by $A \cap B$ (read as 'A intersection B' or 'A cap B') is defined as the set of all elements which belong to both A and B.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

In the following Venn diagram, A and B are the sets represented by the respective circular regions and $A \cap B$ is shown shaded :



$A \cap B$

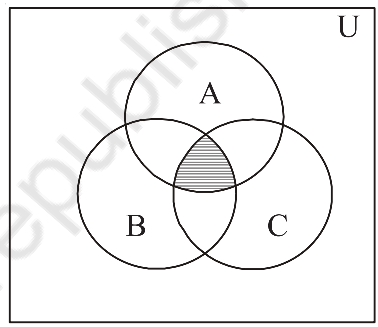
Fig. 1.4

1.20. Intersection is associative :

If A, B and C are any three sets then,

$(A \cap B) \cap C = A \cap (B \cap C)$ .... (i)

Since, each of the sides represents the set of elements which belong to all the three sets A, B and C. Each of these equal sets will be denoted by $A \cap B \cap C$. In figure 1.5, $A \cap B \cap C$ is shown shaded.



$A \cap B \cap C$

Fig. 1.5

The above relation (i) is expressed by saying that set intersection is associative.

1.21. Intersection is commutative :

$A \cap B = B \cap A$ for any two sets A and B ;

because $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= B \cap A$.

Examples : 1. If $A = \{x : x \in Q \text{ and } 1 \leq x \leq 5\}$ and B=Z, the set of integers, then $A \cap B = \{1, 2, 3, 4, 5\}$

2. If A and B are any two sets then $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Proof : If $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular $x \in A$. Thus, each element x in $A \cap B$ is an element of A. This means that $A \cap B$ is a subset of A i.e. $A \cap B \subseteq A$.

Similarly, $A \cap B \subseteq B$.

3. Prove that $A \cap \phi = \phi$ for any set A

   Proof : Clearly $A \cap \phi \subseteq \phi$ (by example 2 above)

   But ϕ is subset of any set. Hence, $\phi \subseteq A \cap \phi$

   Thus the set $A \cap \phi$ and ϕ are subsets of one another.

   Hence, $A \cap \phi = \phi$.

4. If $B \subseteq A$, then prove that $A \cap B = B$,

   Proof : We know that $A \cap B \subseteq B$ .... (i)

Also as $B \subseteq A$, each element in B is also in A. Thus each element in B belongs to both A and B and hence to $A \cap B$. In other words,

$B \subseteq A \cap B$ .... (ii)

From (i) and (ii),

$A \cap B = B$.

5. Prove that for any set A, $A \cap U = A$ where U is the universal set.

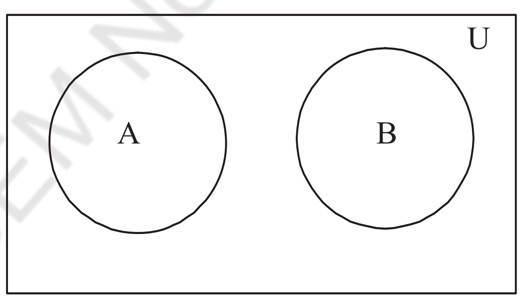
   [i.e. U is the identity with respect to set intersection]

   Proof : If follows from example 4 above and the fact that $A \subseteq U$.

1.22. Disjoint sets :

Definition : Two sets A and B are said to be *disjoint* if $A \cap B = \phi$ i.e. if they have no element in common.

The sets A and B shown in the following Venn diagram are disjoint :



A and B are disjoint

Fig. 1.6

Examples : (i) The sets {1, 2, 3} and {4, 5, 6} are disjoint.

(ii) The set of rational numbers and the set of irrational numbers are disjoint.

(iii) The sets R^+ and R^- are disjoint.

1.23. Intersection of a finite number of sets :

Definition : If $A_1, A_2, A_3, \ldots\ldots\ldots, A_n$ are n sets, then their intersection denoted by $A_1 \cap A_2 \cap \ldots\ldots\ldots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ is by definition the set of all elements belonging to each of the n sets.

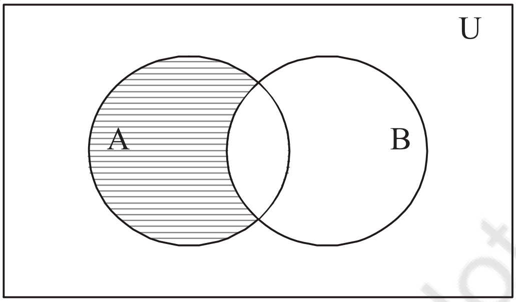
$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \, \forall i \in \{1, 2, 3, \ldots\ldots\ldots, n\}\}$$

Example : If $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{2, 4, 6, 8\}$ and $A_4 = \{2, 5, 8\}$, then $A_1 \cap A_2 \cap A_3 \cap A_4 = \{2\}$

1.24. Difference :

If $A = \{a, b, c, d, e\}$ and $B = \{b, d, f, g\}$ what are the elements of A which do not belong to B ? They are *a, c* and *e*. These elements form a set called A minus B.

Definition : If A and B are two sets ; then the *difference* set denoted by $A-B$ (read as 'A difference B' or A minus B') is defined to be the set of all elements which belong to A but not to B. Thus $A-B = \{x : x \in A$ and $x \notin B\}$. We also sometimes write "A\B" or "A/B" for $A-B$.



A–B
Fig. 1.7

In the adjoining Venn diagram, the shaded portion represents $A-B$.

$A-B$ is also known as the *relative complement* of B in A. It is readily seen that $A-B \subseteq A$. (why ?)

***Example* 1 :** Prove that $A-A = \phi$ for any set A.

Proof : $A-A = \{x : x \in A$ and $x \notin A\}$

$= \phi$, since there is no element x which belongs to A and does not belong to A at the same time.

2. Prove that $A-B$ and B are disjoint.

Proof : It is readily seen from the Venn diagram (fig. 1.7).

[A separate proof can be given as follows :

Suppose $x \in (A-B) \cap B$. Then $x \in A-B$ and $x \in B$. This means that $x \in A$ and $x \notin B$ and $x \in B$. As there is no elements x which belongs to B and does not belong to B at the same time, it follows that

$$(A-B) \cap B \subseteq \phi$$

But, ϕ is a subset of every set.

$$\therefore \phi \subseteq (A-B) \cap B$$

Being subsets of each other, we have $(A-B) \cap B=\phi$ i.e. A–B and B are disjoint]

1.25. Complement :

It has already been stated that all sets that occur in a discussion are subsets of one and same set U called Universal set for the discussion.

Definition : The *complement* of a set A denoted by A' or A^c is defined as the set of all elements which do not belong to A and of course which belong to U.

Thus, $A' = \{x : x \notin A\}$
$= \{x : x \in U \text{ and } x \notin A\}$
$= U - A$

In the adjoining Venn diagram A is represented by the circular region and A' by the shaded region.

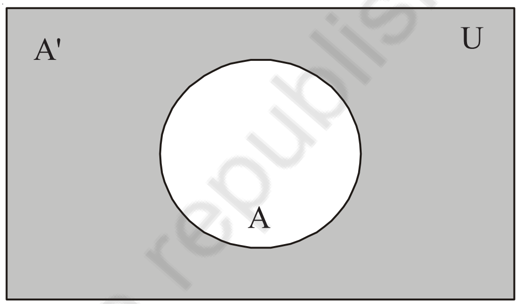


Fig. 1.8

Examples 1. If U={1, 2, 3, 4, 5, 6, 7, 8} and A={1, 3, 5, 7} then A'={2, 4, 6, 8}

2. If R be the universal set, the complement of Q is Q′, the set of all irrational numbers.

1.26. De Morgan's laws of complementation :

These laws can be stated as follows :

If A and B are any two sets, then

(i) $\cup B)'=A' \cap B'$ (i.e. the complement of the union of two sets is the intersection of their complements.)

(ii) $\cap B)'=A' \cup B'$ (i.e. the complement of the intersection of two sets is the union of their complements.)

1.27. Some basic facts about complementation :

Besides De Morgan's laws, there are some simple facts about complementation. They are given below :

(a) $(A')'=A$

(b) (i) $U'=\phi$ and (ii) $\phi'=U$

(c) (i) $A\cap A'=\phi$ and (ii) $A\cup A'=U$

(d) $A\subseteq B\Leftrightarrow B'\subseteq A'$

The validity of these results can be visualised by drawing respective Venn diagrams. (Verify them by Venn Diagrams.)

1.28. Distributive laws :

If A, B, and C are any three sets, then

(i) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

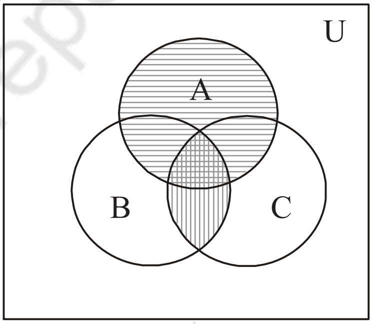
[union distributes over intersection]

(ii) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

[intersection distributes over union]

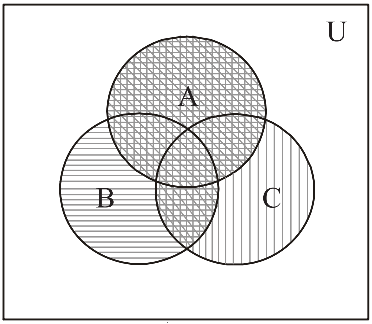
The above law (i) can be illustrated by means of Venn diagrams as follows :

Let us shade the set A by horizontal lines and $B\cap C$ by vertical lines. Then entire shaded region represents $A\cup(B\cap C)$



$A\cup(B\cap C)$
Fig. 1.9

Again, let us shade $A\cup B$ by horizontal lines and $A\cup C$ by vertical lines. Then $(A\cup B)\cap(A\cup C)$ is the portion which has been shaded both ways.



$(A\cup B)\cap(A\cup C)$
Fig. 1.10

Thus, from figures 1.9 and 1.10 it is clear that $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

[Exercise : Illustrate the other by Venn diagrams]

Examples :

1. Prove that

(i) $A-B=A\cap B'$

(ii) $B-A'=A\cap B$

Proof : (i) $A - B = \{x : x \in A \text{ and } x \notin B\}$

$= \{x : x \in A \text{ and } x \in B'\}$

$= A \cap B'$

(ii) $B - A' = \{x : x \in B \text{ and } x \notin A'\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= \{x : x \in A \text{ and } x \in B\}$

$= A \cap B$

2. If A and B are any two sets, prove that $A \cap B$, $A - B$ and $B - A$ are pairwise disjoint.

Proof : We have to prove that each of the three sets $(A \cap B) \cap (A - B)$, $(A \cap B) \cap (B - A)$ and $(A - B) \cap (B - A)$ is empty.

$$(A \cap B) \cap (A - B) = (A \cap B) \cap (A \cap B')$$

$$= A \cap (B \cap A) \cap B' \quad \text{(by associativity of } \cap)$$

$$= A \cap (A \cap B) \cap B' \quad (\because A \cap B = B \cap A)$$

$$= (A \cap A) \cap (B \cap B') \quad (\cap \text{ is associative})$$

$$= A \cap \phi$$

$$= \phi \text{ (Example 3 of § 1.21)}$$

Interchanging the roles of A and B, we obtain from the above

$$(B \cap A) \cap (B - A) = \phi$$

i.e. $(A \cap B) \cap (B - A) = \phi$

Finally, $(A - B) \cap (B - A) = (A \cap B') \cap (B \cap A')$

$$= A \cap (B' \cap B) \cap A'$$

$$= A \cap \phi \cap A'$$

$$= \phi$$

This completes the proof.

The result can also be illustrated by Venn diagram.

Let us shade $A \cap B$ by horizontal lines, $A - B$ by vertical lines, $B - A$ by dots. It is observed that no two of the shaded portions are overlapping. This shows that $A \cap B$, $A - B$ and $B - A$ are pairwise disjoint.

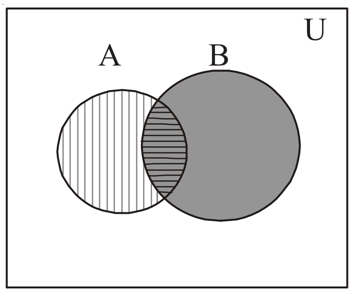


Fig. 1.11

3. Prove that : (i) $A\cap(A'\cup B)=A\cap B$ (ii) $A\cap(A\cup B)=A\cup(A\cap B)=A$

Proof : (i) $A\cap(A'\cup B) = (A\cap A')\cup(A\cap B)$ (distributive law)

$= \phi\cup(A\cap B)$ ($\because A\cap A'=\phi$)

$= A\cap B$ ($\because \phi\cup X=X$)

(ii) $A\cap(A\cup B) = (A\cap A)\cup(A\cap B)$ (distributive law)

$= A\cup(A\cap B)$ (Idempotent law)

$= A$ (since $A\cap B\subseteq A$)

EXERCISE 1.2

1. Fill in the blanks with proper words or symbols :

   (i) If $x\in A\cup B$ then $x\in A$ .... $x\in B$. (ii) If $x\in A\cap B$ then $x\in A$ .... $x\in B$.

   (iii) If $x\notin A\cap B$ then x .... A ..... x ....B. (iv) If $x\in A-B$ then $x\in A$ and x ..... B.

   (v) If $x\in A'\cap B'$ then x .... A ..... x ....B.

2. If $A=\{a, b, c, d, e\}$, $B=\{x, y, z\}$ and $C=\{b, c, p, x, z\}$ find

   (i) $A\cup B$ (ii) $B\cap C$ (iii) $C\cup A'$ (iv) $A\cap B$ (v) $B\cap C$ (vi) $C\cap A$ (vii) $A-B$ (viii) $B-C$ (ix) $C-A$ (x) $B-A$ (xi) $C-B$ (xii) $A-C$ (xiii) $(A\cup B)\cup C$ and $A\cup(B\cup C)$ (xiv) $A\cap(B\cup C)$ (xv) $A\cup(B\cap C)$

3. Let $A=\{1, 2, 5, 7\}$, $B=\{2, 5, 9\}$ and $C=\{5, 7, 8, 9\}$. If the universal set U is the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ find.

   (i) A' (ii) B' (iii) C' (iv) $A'\cap B$ (v) $B\cap C'$

   (vi) $A'\cap C'$ (vii) $(A\cup B)'$ (viii) $(A\cap C)'$ (ix) $B'\cup C'$

   (x) $A'-B'$ (xi) $A'\cup(B\cup C)$ (xii) $A\cap(B'\cup C')$ (xiii) $A'\cup(B'\cup C')$

   (xiv) $(A'\cup B')\cap(A'\cup C')$

4. Give examples to show that the following statements are false :

   (i) $A-B=A\Rightarrow B=\phi$ (ii) $A\cup B=A\Rightarrow B=\phi$ (iii) $A\cap B=\phi\Rightarrow A=\phi$ or $B=\phi$

   (iv) $(A\cup B)-B=A$ (v) $A-B=\phi\Rightarrow A=B$ (vi) $A\cup B=A\cup C\Rightarrow B=C$

   (vii) $A\cap B=A\cap C\Rightarrow B=C$

5. Prove that the following pairs of sets are disjoint (by Venn diagrams)

   (i) $A-B$ and $B-A$ (ii) $A\cap B$ and $A-B$ (iii) $A\cap B$ and $B-A$

   (iv) $A-B$ and B (v) $B-A$ and A

6. If $A\cup B=\phi$, then prove that $A=\phi$ and $B=\phi$.

7. If $A \subseteq S$ and $B \subseteq S$, prove that $A \cup B \subseteq S$.
8. If $A \subseteq B$, prove that $A \cap B = A$ and $A \cup B = B$.
9. If $A \cap B = \phi$, prove that $A \subseteq B'$ and $B \subseteq A'$.
10. If $A \cup B = U$, verify by Venn diagrams that $A' \subseteq B$.
11. If $A \cap B = \phi$, verify by Venn diagrams that $A - B = A$ and $B - A = B$.

1.29. Cardinal numbers of finite sets :

Definition : If A be a finite set, then the number of elements in A, denoted by $n(A)$ is called the *cardinal number* or *cardinality* of A.

If A and B are two disjoint finite sets, then the number of elements in their union will be the sum of the numbers of elements in each of them.

i.e. $n(A \cup B) = n(A) + n(B)$.

Similarly, if A, B and C are pairwise disjoint, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C).$$

We shall now prove the following theorem.

1.30. Theorem 1 :

For any two finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof : We have seen that the sets $A \cap B$ and $A - B$ are disjoint. Now

$$\begin{aligned}
(A \cap B) \cup (A - B) &= (A \cap B) \cup (A \cap B') && [\text{Since } A - B = A \cap B'] \\
&= A \cap (B \cup B') && [\text{by distributive law}] \\
&= A \cap U && [\text{where U is the universal set}] \\
&= A \\
\therefore\ n(A) &= n(A \cap B) + n(A - B) \quad \ldots\ldots\ldots\ldots (1)
\end{aligned}$$

Interchanging roles of A and B, we obtain,

$$n(B) = n(A \cap B) + n(B - A) \quad \ldots\ldots\ldots\ldots (2)$$

Adding (1) and (2),

$$\begin{aligned}
n(A) + n(B) &= 2n(A \cap B) + n(A - B) + n(B - A) \\
\therefore\ n(A) + n(B) - n(A \cap B) &= n(A \cap B) + n(A - B) + n(B - A) \quad \ldots\ldots\ldots\ldots (3)
\end{aligned}$$

The theorem will be established if we show that the right side of equation (3) is the same as $n(A \cup B)$. To show this, let us first prove that

$A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.

This fact is readily visualised from the following Venn diagram :

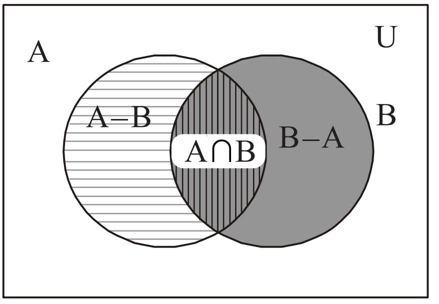


Fig. 1.12

However, formal proof can also be given as follows :

Since $(A\cap B)\cup(A-B)=A$ we have,

$$\begin{aligned}(A\cap B)\cup(A-B)\cup(B-A) &= A\cup(B-A)\\ &= A\cup(B\cap A')\\ &= (A\cup B)\cap(A\cup A') && \text{[distributive law]}\\ &= (A\cup B)\cap U && [\because A\cup A'=U]\\ &= A\cup B && [\because U \text{ is identity for } \cap]\end{aligned}$$

As the three sets $A\cap B$, $A-B$ and $B-A$ are pairwise disjoint, it follows that,

$$n(A\cup B)=n(A\cap B)+n(A-B)+n(B-A) \quad\text{.................................(4)}$$

From (3) and (4) we obtain,

$$n(A\cup B)=n(A)+n(B)-n(A\cap B).$$

1.31. Theorem 2 :

For any three sets, A, B and C,

$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C).$

Proof : $n(A\cup B\cup C)=n(P\cup C)$ where $P=A\cup B$

$= n(P)+n(C)-n(P\cap C)$ [Theorem 1]

$= n(A\cup B)+n(C)-n[(A\cup B)\cap C]$

$= n(A)+n(B)-n(A\cap B)+n(C)-n[(A\cap C)\cup(B\cap C)]$

$= n(A)+n(B)+n(C)-n(A\cap B)-[n(A\cap C)+n(B\cap C)-n\{(A\cap C)\cap(B\cap C)\}]$

$= n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C\cap C)$

$= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C).$

Example :1. In a school examination, 400 students appeared in two papers English and Mathematics ; 333 students passed in English, 315 passed in Mathematics. And 273 passed in both. How many failed in both ?

Solution. Let A be the set of students who failed in English and B be the set of students who failed in Mathematics. Then $A \cup B$ is the set of students who failed in English or Mathematics and $A \cap B$ in the set of students who failed in both.

By the given condition,

$$n(A) = 400 - 333 = 67$$

$$n(B) = 400 - 315 = 85$$

and $$n(A \cup B) = 400 - 273 = 127$$

Since, $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\therefore \quad 127 = 67 + 85 - n(A \cap B)$$

$$\Rightarrow n(A \cap B) = 25$$

i.e. the number of students who failed in both = 25.

2. In a group of 475 students, 400 can speak Manipuri, 250 can speak Hindi, 300 can speak English, 200 can speak both Manipuri and Hindi, 225 can speak both Hindi and English and 295 can speak both English and Manipuri. How many can speak all the three ? Assume that each student can speak at least one the three languages.

Solution. Let $A = \{x : x \text{ can speak Manipuri}\}$

$B = \{x : x \text{ can speak Hindi}\}$

$C = \{x : x \text{ can speak English}\}$.

Then, by the given data

$n(A) = 400$, $n(B) = 250$, $n(C) = 300$, $n(A \cap B) = 200$,

$n(B \cap C) = 225$, $n(C \cap A) = 295$ and $n(A \cup B \cup C) = 475$

Now, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$\therefore 475 = 400 + 250 + 300 - 200 - 225 - 295 + n(A \cap B \cap C)$

$\Rightarrow n(A \cap B \cap C) = 245$

i.e. 245 students can speak all the three languages.

EXERCISE 1.3

1. Out of 50 students in a class, 35 can play hockey and 43 can play football. If each student can play at least one of the two games, find the number of students who can play both.
2. In a school, there are 20 teachers. Out of them only 6 can teach Mathematics and 18 can teach English. If each teacher teaches one or the other subject, find the number of teachers who can teach both.

3. In a certain college, 300 students offer Mathematics, 225 offer Physics, 250 offer Chemistry, 200 offer both Mathematics and Physics, 115 offer Physics and Chemistry and 172 offer Chemistry and Mathematics. If 100 students offer all the three, find the number of students who offer at least one of the three subjects.
4. There are 50 players in a club, each playing at least one of the games badminton, tennis and chess. 30 of them play badminton, 16 play tennis, 8 play both badminton and tennis, 10 play both tennis and chess, 18 play both badminton and chess and 6 play all the three. Find the number of players who play chess.
5. In a flood relief camp of 128 persons, 25 were men and the rest women and children. After a week, 69 left the camp, 35 of them being children. Out of those who remained behind, 14 were men. How many women left the camp ?
6. There are 25 pencils on a table ; some are blue and other black in colour. 16 of them have eraser at one end. Number of blue pencils is 14 and 5 of the black ones do not have eraser. How many blue pencils will have eraser ?
7. There are 100 families in a certain locality. 50 of them use gas and 80 of them use kerosene for cooking. Find the number of families using both gas and kerosene for cooking. (Assume that each family uses either gas or kerosene for cooking.)
8. A survey of class X standard boys of a school about whether they play Football, Hockey or Cricket produced the following table :

F	H	C	$F \cap H$	$H \cap C$	$C \cap F$	$F \cap H \cap C$
60%	50%	50%	30%	20%	30%	10%

   (i) What percentage play Football and Hockey but not Cricket ?

   (ii) What percentage play none of these games ?
9. S is a finite set of positive integers, each of which is divisible by 2 or 5 or 11. Among the elements of S, 195 are multiples of 2, 170 are multiples of 5, 140 are multiples of 11, 80 are multiples of 10, 45 are multiples of 22, 30 are multiples of 55 and 20 are multiples of 110. Find the number of elements in S and find how many of them are divisible by 2 or 5 but not by 11.

1.32. Cartesian Product of two sets :

The elements, say x and y belonging to the same set or different sets form an ordered pair denoted by (x, y). Here x is called the first component or first co-ordinate and y, the second component or second coordinate of the ordered pair. Two different elements x and y can form two different ordered pairs, namely (x, y) and (y, x). It is customary to use the parentheses (first brackets) in forming an ordered pair and a comma to separate the first and second components.

In an ordered pair, the order of the elements forming the pair is taken into account and $(x, y) \neq (y, x)$ unless $x = y$.

Definition : Let A and B be any two sets. Let us form ordered pairs with elements of A as first components and those of B as second components. The collection of all such ordered pairs is called the *cartesian product* of A and B, and it is denoted by A×B.

Thus, in set-builder notation

$$A\times B = \{(x, y) : x\in A,\ y\in B\}$$

The definition will become evident if we observe the following examples.

Example 1 : Let $A=\{1, 2\}$ and $B=\{4, 5, 6\}$. Then

$$A\times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)\}$$
$$B\times A = \{(4, 1), (5, 1), (6, 1), (4, 2), (5, 2), (6, 2)\}$$
$$A\times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$
$$B\times B = \{(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$$

Example 2 : If R is the set of all real numbers, then R×R (also denoted by R^2) is $\{(x, y) : x, y\in R\}$ which is the Cartesian co-ordinate plane ; for an element (x, y) of the product forms the co-ordinates of a point in the plane and every point in the plane gives an element of the product viz. its co-ordinates.

From the above example 1, it is clear that if A has 2 elements and B, 3 elements, then both A×B and B×A have $2\times 3=6$ elements. Generalising this result, it is seen that if A has m elements and B, n elements then A×B (also B×A) has mn elements.

This fact implies that

(i) A×B is an infinite set if either A or B or both are infinite

and (ii) $A\times B=\phi$ if either A or B or both are empty.

Example 3 : If $A\subseteq X$ and $B\subseteq Y$, prove that $A\times B\subseteq X\times Y$.

Solution : Suppose (a, b) is an element of A×B. Then $a\in A$ and $b\in B$.

But $A\subseteq X$; so $a\in A\Rightarrow a\in X$. Similarly, $b\in B\Rightarrow b\in Y$. It follows that $(a, b)\in X\times Y$.

Thus, any element (a, b) of A×B is in X×Y. Hence $A\times B\subseteq X\times Y$.

Example 4 : If A×B contains 9 elements of which three are (1, 1), (2, 3) and (3, 4), write down all the nine elements.

Solution : The possible numbers of elements in A and in B are

(i) $n(A)=1$ and $n(B)=9$

(ii) $n(A)=3$ and $n(B)=3$

(iii) $n(A)=9$ and $n(B)=1$ $[\because\ 9=1\times 9=3\times 3=9\times 1]$

Since the three elements 1, 2 and 3 belong to A (why ?), therefore $n(A) \geq 3$. Similar reasoning gives $n(B) \geq 3$. So, out of the above possibilities, we must have (ii) i.e. $n(A) = 3$ and $n(B) = 3$.

Then it follows that

$$A = \{1, 2, 3\} \text{ and } B = \{1, 3, 4\}$$

Hence, the elements of A×B are (1,1), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,3) and (3,4).

1.33. Relations :

Let A and B be two given sets.

A *relation* from A to B is defined as a subset of A×B.

Let R be a relation from A to B so that $R \subseteq A \times B$. For $a \in A$ and $b \in B$ if $(a, b) \in R$ then we write aRb and say that '*a* is R-related to *b*'. And if $(a, b) \notin R$, then we write $a \not R b$ or $a \sim Rb$ and we say that '*a* is not R-related to *b*'.

For any two sets A and B, $\phi \subseteq A \times B$ so that ϕ is a relation from A to B. This relation ϕ is known as void relation and it contains no elements to discuss about. Some authors don't consider it as a relation.

Examples :

(i) If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$ then

$R = \{(1, c), (1, e), (2, a), (3, d)\}$ is a relation from A to B.

Here, $1Rc$ for $(1, c) \in R$ and $2 \not R b$ for $(2, b) \notin R$.

(ii) Let $R = \{(n, x) : n \in N, x \in Q \text{ and } nx = 1\}$. Then R is a relation from N to Q. Here, any natural number n is R-related to its reciprocal $\frac{1}{n}$ which is a rational number i.e. $nR\frac{1}{n}\ \forall n \in N$, however $n \not R x$ for any rational number x other than $\frac{1}{n}$.

1.34. Domain and Range of a relation :

Let A and B be any two sets and R be a relation from A to B. Then the set of the first components of the elements of R i.e. the set $\{x : (x, y) \in R \text{ for some } y \in B\}$ is called the *domain* of R.

And the set of the second components of the elements of R i.e. the set $\{y : (x, y) \in R \text{ for some } x \in A\}$ is called the *range* of R.

Clearly, the domain of R is a subset of A and the range of R is a subset of B.

For the relation R given in the above example (i), the domain is {1, 2, 3} i.e. A and the range is $\{a, c, d, e\}$ which is a subset of B. And for the relation in example (ii), the domain is N and the range is $\{\frac{1}{n} : n \in N\}$ which is a subset of Q.

1.35. Inverse Relation :

Let R be a relation from a set A to a set B. The relation R^{-1} from B to A given by $R^{-1}=\{(b, a) : (a, b)\in R\}$ is said to be the *inverse relation* of R.

It is clear that $aRb \Leftrightarrow bR^{-1}a$ i.e. $(a, b)\in R \Leftrightarrow (b, a)\in R^{-1}$. And for every relation R, the inverse relation R^{-1} exists.

Consider the relation $R=\{(a, l), (a, m), (b, l), (c, n)\}$ from the set $A=\{a, b, c\}$ to the set $B=\{l, m, n\}$. Then the relation $R^{-1}=\{(l, a), (m, a), (l, b), (n, c)\}$ from B to A is the inverse relation of R.

1.36. Relation in a set and Types of Relations :

Let A be a set. Then a relation from A to A i.e. a subset of A×A is called a *relation* in A.

Examples :

(i) $R_1=\{(1, 1), (1, 2)\ (2, 1), (2, 2), (2, 3)\}$ is a relation in the set $A=\{1, 2, 3\}$

(ii) $R_2=\{(x,y) : x, y\in R\ ;\ x^2+y^2=1\}$ is a relation in the set R of real numbers. Here R_2 contains infinite number of ordered pairs.

(iii) $R_3=\{(m, n) : m, n\in N \text{ and } 2n+3m\leq 10\}$ is a relation in the set N of natural numbers. R_3 has only a finite number of elements namely (1, 1), (1, 2), (2, 1), (2, 2) and (3, 1).

Let us now consider some special types of relations which are of frequent use in various branches of pure mathematics.

I. Reflexive Relation : A relation R in a set A is said to be reflexive if each element of A is R-related to itself i.e. $aRa\ \forall a\in A$.

Thus, a relation R in A is reflexive if and only if the set $\{(a, a) : a\in A\}$ is a subset of R.

Example : The relation R={(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)} in the set A={1, 2, 3} is reflexive whereas S={(1, 1), (2, 3), (3, 1), (3, 3)} is not a reflexive relation in A for $2\in A$ but $(2, 2)\notin S$.

Note : A relation R in a set A is not reflexive if and only if there exists at least one element, say x in A such that $x \not{R} x$.

II. Symmetric Relation : A relation R in a set A is said to be symmetric if whenever a is R-related to b, b is R-related to a i.e. $aRb\Rightarrow bRa$.

Examples :

(i) If $A=\{a, b, c, d\}$ and $R=\{(a, b), (b, a), (b, c), (c, b), (d, d)\}$, then R is a symmetric relation in A, for there are no elements x, y in A such that xRy but $y \not{R} x$.

(ii) If $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$, then R as a relation in A is not symmetric for 1R2 but $2 \not R 1$.

Note : A relation R in set A is not symmetric if and only if there are elements, say a and b in A such that aRb but $b \not R a$. And R is symmetric if and only if the inverse relation R^{-1} coincides with R i.e. $R^{-1} = R$.

III. Transitive Relation : A relation R in a set A is said to be transitive if whenever a is R-related to b and b to c, a is R-related to c i.e. if aRb, $bRc \Rightarrow aRc$.

Example : The relation $R_1 = \{(a, b), (b, c), (c, d)\}$ in the set $A = \{a, b, c, d\}$ is not transitive, for aR_1b and bR_1c but $a \not R_1 c$. And the relation $R_2 = \{(a, a), (a, b), (b, a), (c, d)\}$ in A, is transitive for we cannot find elements, say x, y, z in A such that xR_2y, yR_2z but $x \not R_2 z$.

Note : A relation R in a set A is not transitive if and only if there are elements x, y, z (say) in A such that xRy, yRz but $x \not R z$.

IV. Anti-Symmetric Relation : A relation R in a set A is said to be anti-symmetric if aRb and $bRa \Rightarrow a=b$.

Example : Consider the following relations in the set $A = \{1, 2, 3\}$

$$R_1 = \{(1, 2), (2, 3), (3, 1)\}$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\}$$
$$R_3 = \{(1, 1), (2, 2), (3, 3)\}$$

Here, R_1 is anti-symmetric but not symmetric, R_2 is symmetric but not anti-symmetric and R_3 is symmetric as well as anti-symmetric.

V. Identity Relation : A relation R in a set A is said to be identity relation if $R = \{(a, a) : a \in A\}$.

Given the set A, identity relation in A is unique and it is usually denoted by I_A. If $A = \{1, 2, 3\}$ then $I_A = \{(1, 1), (2, 2), (3, 3)\}$.

For any set A, the identity relation I_A is reflexive, symmetric, transitive as well as anti-symmetric.

VI. Universal Relation : For any set A, the product set $A \times A$ is a relation in A. This relation is called the *universal relation* in A.

If R denotes the universal relation in a set A, then each element $a \in A$ is R-related to every element $b \in A$. Also R is reflexive, symmetric and transitive.

VII. Equivalence Relation : A relation in a set is said to be an equivalence relation if it is reflexive, symmetric and transitive.

For a given set A, both the identity relation and the universal relation in A are equivalence relations.

Example 1 : A relation R is defined in the set of real numbers by $x\text{R}y$ *iff* $|x|=|y|$. Show that it is an equivalence relation. Is it anti-symmetric ?

Solution : For any real number x, we have $|x|=|x|$ i.e. $x\text{R}x$.

So, R is reflexive.

Suppose $x\text{R}y$, where x, y are real numbers. Then

$$|x|=|y|$$
$$\Rightarrow |y|=|x|$$
$$\Rightarrow y\text{R}x$$

Thus, $x\text{R}y \Rightarrow y\text{R}x$ and hence R is symmetric.

Again, for real number, x, y and z,

$$x\text{R}y,\ y\text{R}z \Rightarrow |x|=|y| \text{ and } |y|=|z|$$
$$\Rightarrow |x|=|z|$$
$$\Rightarrow x\text{R}z$$

$\therefore$ R is transitive.

It follows that R is an equivalence relation.

Clearly, R is not anti-symmetric, for (2, –2) and (–2, 2) both belong to R but $2\neq-2$ i.e. $2\text{R}-2$ and $-2\text{R}2$ but $2\neq-2$

Example 2 : In the set N of natural numbers, a relation R is defined by $m\text{R}n \Leftrightarrow m$ is a factor of n. Examine if R is (i) Reflexive (ii) Symmetric (ii Anti-symmetric (iv) Transitive.

Solution : (i) For each $m\in$ N, we know that m is a factor of m i.e. $m\text{R}m$. Hence, R is reflexive.

(ii) R is not symmetric, for 2, $4\in$ N and 2 is a factor of 4 but 4 is not a factor of 2 i.e. 2R4 but $4\not\text{R}\,2$.

(iii) For m, $n\in$ N, if m is a factor of n and n is a factor of m then clearly m should be equal to n. In other words, $m\text{R}n$ and $n\text{R}m \Rightarrow m=n$. Hence, R is anti-symmetric.

(iv) If m is a factor of n and n, a factor of p, then clearly m is a factor of p i.e. $m\text{R}n$, $n\text{R}p \Rightarrow m\text{R}p$. Hence R is transitive.

Example 3 : In the set Q of rational numbers, a relation R is defined by $x\text{R}y \Leftrightarrow 1+xy>0$. Show that R is reflexive and symmetric but not transitive.

Solution : For any $x\in$ Q, we have $1+x.x=1+x^2>0$

$\therefore x\text{R}x \forall x\in$ Q and so R is reflexive.

Again, for $x, y \in Q$

$$x R y \Rightarrow 1+xy>0$$
$$\Rightarrow 1+yx>0 \quad [\because xy=yx]$$
$$\Rightarrow y R x \quad [\text{by definition of R}]$$

$\therefore$ R is symmetric.

That R is not transitive can be verified by considering the rational numbers $x=\frac{1}{2}$, $y=-1$ and $z=-2$. Here,

$$1+xy = 1+\left(-\frac{1}{2}\right) = \frac{1}{2}>0 \quad \text{and so } xRy,$$
$$1+yz = 1+2 = 3>0 \quad \text{ans so } yRz,$$
$$1+xz = 1+(-1) = 0 \not> 0 \quad \text{and so } x \not R z.$$

Thus, there are rational numbers x, y, z such that xRy and yRz but $x \not R z$ and so R is not transitive.

Example 4 : Given the relation {(1, 1), (1, 2)} in the set {1, 2, 3}, find the minimum number of ordered pairs to be added so that the enlarged relation is an equivalence relation.

Solution : Let A = {1, 2, 3} and R_1 = {(1, 1), (1, 2)}. Clearly R_1 as a relation in A is not reflexive. To make it reflexive, the ordered pairs (2, 2) and (3, 3) must be added at least. And to make the relation symmetric the ordered pair (2, 1) must also be added.

Let us now check whether the enlarged relation, say

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

is an equivalence relation or not.

It is easily seen that R is reflexive and symmetric. Also, we cannot find elements $a, b, c \in A$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$. Hence, R is transitive.

[In fact, the pairs of elements of R in which second component of the first and the first component of the second are equal, are only the following :

(i) (1, 1), (1, 2)
(ii) (1, 2), (2, 1)
(iii) (1, 2), (2, 2)
(iv) (2, 1), (1, 2)
(v) (2, 2), (2, 1)
(vi) (2, 1), (1, 1)

In each case, the ordered pair formed by taking the first component of the first and the second component of the second, is again a member of R].

It follows that R is an equivalence relation in A. Thus, the minimum number of ordered pairs to be added is 3 and the ordered pairs to be added are (2, 1), (2, 2) and (3, 3).

Example 5 : Prove that the relation R defined in Z by $a\text{R}b$ *iff* $a-b$ is divisible by 5 ($a, b \in \text{Z}$), is an equivalence relation.

Solution : For any integer a, we know that $a-a=0$, which is divisible by 5.

So, $a\text{R}a \,\forall a \in \text{Z}$ and consequently R is reflexive.

Suppose $a\text{R}b$ where $a, b \in \text{Z}$. Then $a-b$ is divisible by 5.

$\therefore\ -(a-b)$ i.e. $b-a$ is also divisible by 5.

$\therefore\ b\text{R}a$ (by definition of R).

Thus, $a\text{R}b \Rightarrow b\text{R}a$ and consequently R is symmetric.

Next, suppose $a\text{R}b$ and $b\text{R}c$; then both $a-b$ and $b-c$ are divisible by 5. So, $(a-b)+(b-c)$ i.e. $a-c$ is also divisible by 5. This means that $a\text{R}c$.

Thus, $a\text{R}c$ whenever $a\text{R}b$ and $b\text{R}c$. Hence, R is transitive.

It follows that R is an equivalence relation in Z.

Example 6 : If R and S are equivalence relations in a set A, prove that $\text{R} \cap \text{S}$ is also an equivalence relation in A. Verify by means of an example that $\text{R} \cup \text{S}$ need not be an equivalence relation.

Solution : For every $a \in \text{A}$, $(a, a) \in \text{R}$ and $(a, a) \in \text{S}$ ($\because$ both R and S are reflexive). Hence $(a, a) \in \text{R} \cap \text{S}$, $\forall a \in \text{A}$ i.e. $\text{R} \cap \text{S}$ is reflexive.

If $(a, b) \in \text{R} \cap \text{S}$, then $(a, b) \in \text{R}$ and $(a, b) \in \text{S}$. But,

$(a, b) \in \text{R} \Rightarrow (b, a) \in \text{R}$

and $(a, b) \in \text{S} \Rightarrow (b, a) \in \text{S}$. ($\because$ both R and S are symmetric)

$\therefore\ (b, a) \in \text{R} \cap \text{S}$

Thus, $(a, b) \in \text{R} \cap \text{S} \Rightarrow (b, a) \in \text{R} \cap \text{S}$ and so $\text{R} \cap \text{S}$ is symmetric.

Again, suppose $(a, b), (b, c) \in \text{R} \cap \text{S}$. Then (a, b) and (b, c) belong to both R and S. But,

$(a, b), (b, c) \in \text{R} \Rightarrow (a, c) \in \text{R}$ ($\because$ R is transitive)

Similarly, $(a, b), (b, c) \in \text{S} \Rightarrow (a, c) \in \text{S}$

$\therefore\ (a, c) \in \text{R} \cap \text{S}$

Thus $(a, b), (b, c) \in \text{R} \cap \text{S} \Rightarrow (a, c) \in \text{R} \cap \text{S}$ and so $\text{R} \cap \text{S}$ is transitive.

It now follows that $\text{R} \cap \text{S}$ is an equivalence relation.

That $\text{R} \cup \text{S}$ need not be an equivalence relation is verified by the following example :

Let $\text{A} = \{1, 2, 3\}$

Then $\text{R} = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

and $\text{S} = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$

are equivalence relations in A. But

$\text{R} \cup \text{S} = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (2, 1), (3, 2)\}$

is not an equivalence relation for $(1, 2), (2, 3) \in \text{R} \cup \text{S}$ but $(1, 3) \notin \text{R} \cup \text{S}$.

EXERCISE 1.4

1. Find A×A, A×B, B×A and B×B when
   (i) A = {1, 2} and B = {2, 3, 4}
   (ii) A = {a, b, c} and B = {e, f, g}
   (iii) A = {a, b, c, d} and B = {2, 5}.
2. If A×B has six elements of which three are (1, 1), (1, 2) and (3, 3), find the other three elements of A×B. Also find B×A.
3. If A×B has eight elements five of which are (a, a), (a, c), (b, b), (b, c) and (b, d), find the other three elements.
4. Define a relation from a set to a set, its domain and range.
5. Write down all elements of the relation R in the set N given by

   $R = \{(x, y) : x, y \in N, 2x + y = 12\}$

   Find its domain and range.
6. When is a relation R in a set A said to be
   (i) reflexive (ii) symmetric (iii) transitive (iv) anti-symmetric ?
7. Can a relation R in a set A be symmetric as well as anti-symmetric ? Justify your answer by means of an example.
8. Show that the relation "less than or equal to" denoted by ≤, in N, the set of natural numbers is reflexive, anti-symmetric and transitive.
9. Give an example of a relation which is
   (a) symmetric and reflexive but not transitive.
   (b) symmetric and transitive but not reflexive.
   (c) reflexive and transitive but not symmetric.
   (d) reflexive but neither symmetric nor transitive.
   (e) symmetric but neither reflexive nor transitive.
   (f) transitive but neither reflexive nor symmetric.
10. Let A = {1, 2, 3} and R = {(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)}. Prove that R is an equivalence relation in A.
11. Let A be the set of all triangles in a plane and R be the relation in A defined by xRy *iff* x is congruent to y ($x, y \in$ A). Prove that R is an equivalence relation.

12. Let R be the relation in Z defined by $a\text{R}b$ *iff* a is a multiple of b. Prove that R is reflexive, anti-symmetric and transitive.
13. Define an equivalence relation. If R is the relation in the set Z of all integers defined by the open statement "$x-y$ is divisible by 7", that is $\text{R}=\{(x, y) : (x, y) : x\in \text{Z}, y\in \text{Z}, x-y$ is divisible by $7\}$, prove that R is an equivalence relation.
14. If R is an equivalence relation in a set A, prove that the inverse relation R^{-1} is also an equivalence relation in A.
15. Examine if the relation R defined in Q by $x\text{R}y \Leftrightarrow |x-y|<5$ is an equivalence relation or not.

1.37. Functions or Mappings :

If A and B are any two non-empty sets, then a function f from A to B is defined as a relation from A to B i.e. a subset of A×B, satisfying the following two conditions :

(i) $\forall a\in \text{A}$, there exists $b\in \text{B}$ such that $(a, b)\in f$.

(ii) $a, b)\in f$ and $(a, c)\in f \Rightarrow b=c$.

The first condition specifies the function f as a rule which assigns to each element $a\in \text{A}$ one element $b\in \text{B}$. The second condition states that the element $b\in \text{B}$ which is assigned to a given element $a\in \text{A}$ is unique i.e. one and only one. Keeping these two conditions in mind, an alternative definition of function may be given as follows :

A function f from a set A to a set B is a rule which assigns to each element $a\in \text{A}$ a unique element $b\in \text{B}$.

Let $\text{A}=\{1, 2, 3\}$ and $\text{B}=\{a, b, c\}$. Then $f=\{(1, a), (2, b), (3, a)\}$ is a function from A to B, for

(i) each element of A appears as the first component of some ordered pair in f, and

(ii) no two different ordered pairs in f have the same first component. However, $g=\{(1, a), (1, b), (3, c)\}$ is not a function from A to B for the following two reasons :

(1) There is no ordered pair in g with $2\in \text{A}$ as the first component i.e. g does not assign any element of B to the element 2 of A.

(2) The different ordered pairs $(1, a)$ and $(1, b)$ in g, have the same first component $1\in \text{A}$ i.e. g does not assign a unique element to the element $1\in \text{A}$.

We use the symbol $f : \text{A}\rightarrow \text{B}$ to denote that f is a function from A to B.

1.38. Domain, Co-domain and Range of a function :

If f is a function from A to B, then the set A is called the *domain* and B, the *co-domain* of the function f. The element $b \in$ B, which is assigned to a given element $a \in$ A by f, is called the $f-$image of the element a and denoted as $f(a)$. Also then the element $a \in$ A is referred to as a pre-image of $f(a)$ i.e. b. The set of all f-images of elements of A i.e. the set $\{f(x) : x \in A\}$ is called the range of f and is usually denoted by $f(A)$.

Since the f- image of every element of A belongs to B, it follows that the range of f is a subset of the co-domain B.

1.39. Diagrammatic Representation of Mappings :

A mapping $f : A \rightarrow B$ may be represented by a diagram drawn in the following way :

(1) The elements of A and B are represented by points in the interior of two disjoint circles or any two disjoint closed curves.

(2) Each point representing an element of A, is joined by an arc or a line segment (with an arrow head) to the point representing the corresponding image in B.

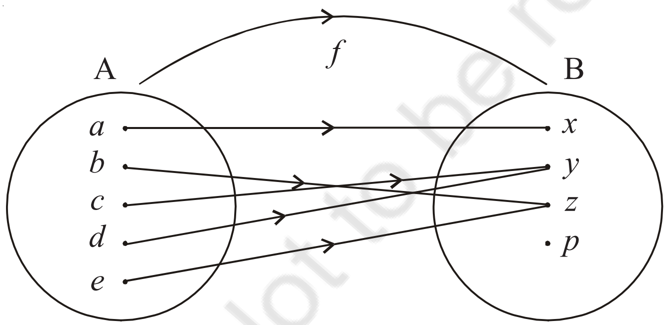


Fig. 1.13

The above diagram (*Fig.* 1.13) represents the mapping $f : A \rightarrow B$ where $A = \{a, b, c, d, e\}$, $B = \{x, y, z, p\}$ and $f(a) = x$, $f(b) = z$, $f(c) = y$, $f(d) = y$, $f(e) = z$. The range of the function is $\{x, y, z\}$.

Note : In a diagrammatic representation of a function $f : A \rightarrow B$

(i) every point of A is joined to some point of B, indicating that every element in the domain has an image in the co-domain,

(ii) a point of A cannot be joined to two or more points of B, indicating that every element in the domain has unique image,

(iii) two or more points of A may be joined to the same point of B, showing that different elements in A may have the same image in B,

(iv) there may be some points of B which are not joined to any points of A i.e. there may be elements in the co-domain which are not images.

1.40. Types of Mappings :

One-one and Many-one Mappings

A mapping $f: A \to B$ is said to be *one to one* (or simply one-one) or injective if different elements in A have different f-images in B.

In an injective mapping, two different elements of the domain cannot have the same image. Thus $f: A \to B$ is injective if

$$x,\ y \in A,\ x \neq y \ \Rightarrow\ f(x) \neq f(y)$$

or equivalently, $x,\ y \in A,\ f(x) = f(y) \ \Rightarrow\ x = y$.

A mapping is said to be *many-one* if it is not one-one.

Thus, a mapping $f: A \to B$ is many-one if and only if there exist at least two distinct elements, say x and y in A such that $f(x) = f(y)$.

Onto and Into Mappings

A mapping $f: A \to B$ is said to be *onto* or *surjective* if each element in B is an f-image of at least one element in A.

Thus, a function $f: A \to B$ is onto if its range coincides with the co-domain i.e. if $f(A) = B$, so that given any element $y \in B$ there must exist some $x \in A$ and $f(x) = y$.

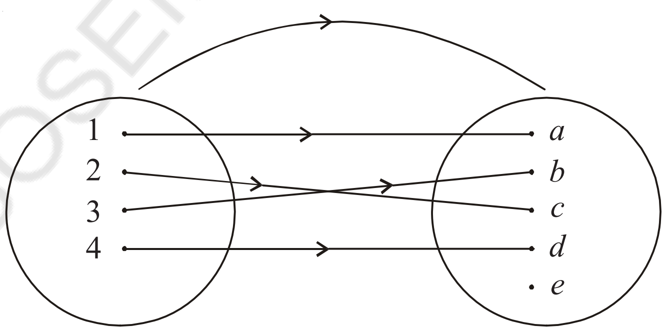
A mapping is said to be *into* if it is not onto.

If $f: A \to B$ is an into mapping, then there is at least one element of B which is not f-image of any element in A.

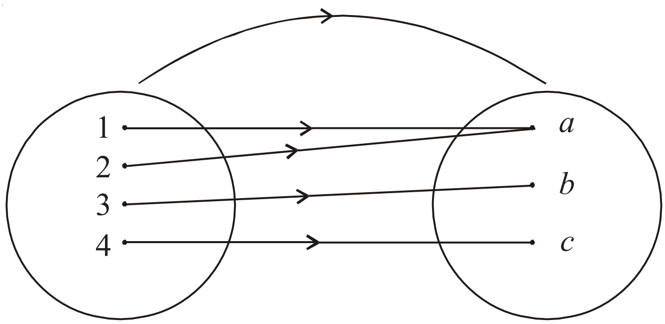
A mapping is said to be *one-one onto* or *bijective* if it is one-one (or injective) as well as onto (or surjective).

A bijective mapping $f: A \to B$ is also called a *bijection* or a *one-one correspondence* between the sets A and B.

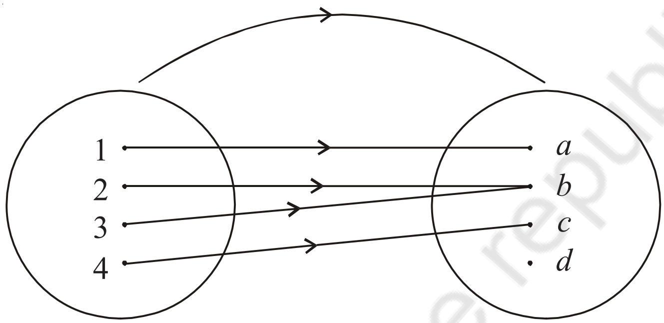
In the following figures, four different types of mapping are described diagrammatically



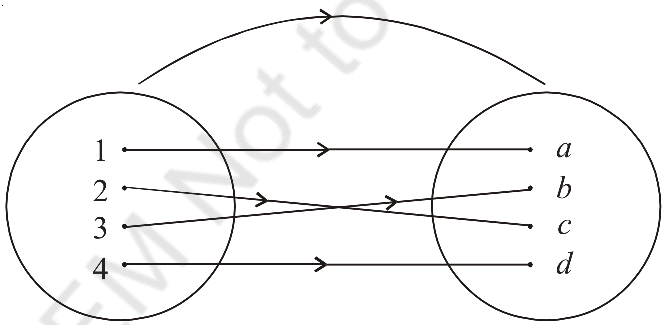
One-one into mapping



Many-one onto mapping



Many-one into mapping



One-one onto mapping

Fig. 1.14

Inclusion mapping and Identity mapping

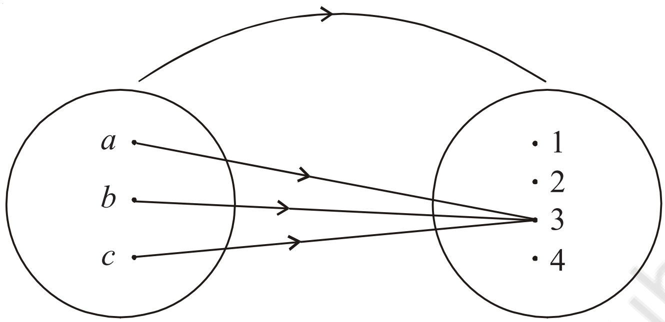
Let A and B be two sets such that $A \subseteq B$. Then $f : A \rightarrow B$ given by $f(x) = x \, \forall x \in A$, is called *inclusion mapping* of A into B. An inclusion mapping is usually denoted by i instead of f.

The inclusion mapping of A into itself is called the *identity mapping* on A and is denoted by I_A. Thus the identity mapping $I_A : A \rightarrow A$ is given by $I_A(x) = x, \ \forall x \in A$. Clearly the identity mapping on A coincides with the identity relation in A.

Constant Function (or Mapping)

A mapping $f: A \rightarrow B$ is called a constant function if its range consists of exactly one element of B i.e. if $f(A) = \{y\}$, for some $y \in B$.

The mapping 'f' represented by the following diagram is a constant function



Example 1 : Can the following collection define a function ? If so, state its domain and range : {(1, 3), (2, 6), (3, 2), (4, 1), (5, 5), (6, 2)}

Solution : Since no two different ordered pairs in the given collection have the same first component, therefore the collection defines a function.

And its

domain = set of all first components = {1, 2, 3, 4, 5, 6}

and range = set of all second components = {1, 2, 3, 5, 6}

Example 2 : If A={1, 2, 3, 4, 5} and f={(1, 2), (2, 3), (3, 2), (4, 1), (5, 1)} , examine if f is a function from A to A. If so, what type of function is it ?

Solution : For each element in A, there is a corresponding ordered pair in f with the element as the first component. Also no two ordered pairs have the same first component. Further, the second components of the elements of f are also elements of A. Hence, f is a function from A to A.

It is seen that the elements 1 and 3 have the same image 2 and there are elements such as 4 and 5 which are not f-images. Therefore $f: A \rightarrow A$ is neither one-one nor onto. Hence, it is many-one and into.

Example 3 : Show that for any non-empty set A, the identity mappy I_A is a bijection.

Solution : The identity mapping $I_A : A \rightarrow A$ is defined by $I_A(x) = x, \forall x \in A$.

Now, $I_A(x) = I_A(y) \quad (x, y \in A) \Rightarrow x = y$

$\therefore$ I_A is injective (or one-one).

Again given any element, say $x \in A$, it is the I_A-image of itself i.e. $I_A(x) = x$. Hence, all elements of A are images and consequently I_A is surjective (or onto).

It follows that I_A is a bijection.

Example 4 : Show that the function $f: R \to R$ defined by $f(x)=x^2$, $\forall x \in R$ is neither one-one nor onto whereas the function $g: R^+ \to R^+$ defined by $g(x)=x^2$, $\forall x \in R^+$ is one-one as well as onto.

Solution : The different elements 1 and -1 in R have the same f-image $1 \in R$, and hence f is not one-one. Also, there are elements such as -2, -3 etc. in the co-domain R which are not f-images and as such f is not onto.

For $x, y \in R^+$, $g(x)=g(y) \Rightarrow x^2=y^2$

$\Rightarrow x^2-y^2=0$

$\Rightarrow (x-y)(x+y)=0$

$\Rightarrow x-y=0$ (since $x+y$ is +ve being sum of positive numbers)

$\Rightarrow x=y$

$\therefore$ g is one-one.

Again, if y is an element of the co-domain R^+, then there exists a positive real number x say, such that $x=\sqrt{y}$ and $g(x)=x^2=\left(\sqrt{y}\right)^2=y$. Thus, every element in R^+ is a g-image and as such g is onto.

It follows that g is one-one as well as onto.

Example 5 : Let $A=\{a, b\}$ and $B=\{a, b, c\}$. How many different functions can be defined from A to B ? Describe all of them as sets of ordered pairs. Identify the constant mappings and the inclusion mapping.

Solution : The different functions from A to B are as described below :

$f_1=\{(a, a), (b, a)\}$, $f_2=\{(a, b), (b, b)\}$, $f_3=\{(a, c), (b, c)\}$

$f_4=\{(a, a), (b, b)\}$, $f_5=\{(a, b), (b, c)\}$, $f_6=\{(a, c), (b, a)\}$

$f_7=\{(a, a), (b, c)\}$, $f_8=\{(a, b), (b, a)\}$, $f_9=\{(a, c), (b, b)\}$

There are therefore, 9 different functions from A to B.

The functions f_1, f_2, f_3 are constant functions or constant mappings for their ranges are singleton sets $\{a\}$, $\{b\}$, $\{c\}$ respectively. The mapping f_4 is the inculsion mapping of A into B (where $A \subseteq B$).

EXERCISE 1.5

1. Which of the following collections define a function ? State domain and range in the case of a function.

(i) $\{(1, a), (2, b), (3, c), (4, a)\}$

(ii) $\{(a, 1), (b, 2), (c, 3), (a, 4)\}$

(iii) $\{(1, 1), (2, 3), (3, 2), (4, 1), (5, 2), (6, 3)\}$

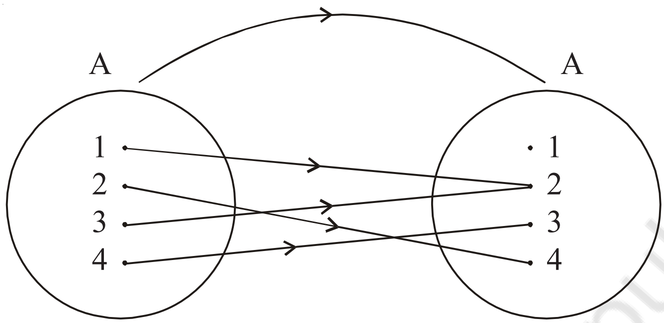
(iv) $\{(1, 1), (2, 2), (3, 1), (4, 2), (2, 3), (1, 2)\}$

2. A function f: R$\rightarrow$R is given by

$$f(x) = \begin{cases} 1, \text{if } x \text{ is rational} \\ 0, \text{if } x \text{ is irrational} \end{cases}$$

Define f in words and find the values of $f(1)$, $f(2)$, $f\left(\frac{1}{3}\right)$, $f\left(\sqrt{2}\right)$ and $f\left(2+\sqrt{3}\right)$.

3. A function f is described by the diagram :



Find the domain, co-domain and range of f. Is the function (i) one-one (ii) onto ?

4. If A = {1, 2} and B = {a, b}, write down all possible functions from A to B as set of ordered pairs. Identify the bijective ones.

5. Let A = {1, 2, 3, 4, 5} and B be the set of positive integers less than 10. A function f:A$\rightarrow$B is given by $f(1)=1$, $f(2)=3$, $f(3)=5$, $f(4)=7$, $f(5)=9$. Give a general formula which can describe f. Examine if f is injective, surjective or bijective.

6. Show that the mapping f: N$\rightarrow$Q defined by $f(n)=\frac{1}{n}$, $n\in$N is one-one but not onto.

7. Show that the function ϕ : Z$\rightarrow$Z defined by $\phi(x)=2x+3$ ($x\in$Z) is one-one but not onto. Find the range of ϕ.

8. Show that the function ϕ : Q$\rightarrow$Q defined by $\phi(x)=2x+3$ ($x\in$Q) is a bijection. Find ϕ-image of 5 and pre-image of 5.

9. Examine whether the following functions are one-one or many-one :

(i) f: N$\rightarrow$N defined by $f(n)=n^2$.

(ii) f: Z$\rightarrow$Z defined by $f(a)=a^2$.

(iii) f: Z$\rightarrow$Z defined by $f(a)=a^3$.

(iv) f: R$\rightarrow$R defined by $f(x)=3x+5$.

(v) f: $R^+\rightarrow R^+$ defined by $f(x)=\frac{1}{x}$.

10. Examine whether the following functions are onto or into :

(i) $f: N \rightarrow N$ defined by $f(n)=n+5$.

(ii) $f: Z \rightarrow Z$ defined by $f(a)=a+5$.

(iii) $f: R^+ \rightarrow R^+$ defined by $f(x)=x^2+5$.

(iv) $f: R \rightarrow R$ defined by $f(x)=3x+5$.

(v) $f: R^+ \rightarrow R^+$ defined by $f(x)=\frac{1}{x}$.

11. Find the range of the function $f: N \rightarrow Q$ defined by

$$f(n)=\begin{cases} n^2, & \text{if } n \text{ is odd} \\ \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

Is the function one-one or many-one ? Give reason for you answer. Find the value of $f(3)$, $f(4)$, $f(5)$, and $f(16)$.

ANSWERS

EXERCISE 1.1

1. Yes, (Refer text for example)

3. (i) $\{1, 2, 4, 5, 10, 20\}$ (ii) $\{c, o, l, e, t, i, n\}$

(iii) $\{2, 3, 5$ $1, 12, 17, 19\}$

4. (i) $\{x : x$ is a digit in the number $2569\}$ (ii) $\{x : x$ is a positive interger less than $8\}$

(iii) $x : x$ is a positive even integer less than $9\}$

(iv) $x : x$ is a positive intergral multiple of $3\}$ or $\{x : x = 3n, \ n \in \mathrm{N}\}$

(v) $x : x$ is a letter of the word "Factor"$\}$

5. (i) F (ii) F (iii) F (iv) T (v) F (vi) T

(vii) F (viii) T (ix) F (x) T (xi) F (xii) T

(xiii) F (xiv) T (xv) T (xvi) F (xvii) T (xviii) F

(xix) False, ϕ has no proper subset (xx) False, e.g. ϕ has only one subset (xxi) T.

6. (i) Finite (ii) Finite (iii) Infinite (iv) Infinite

(v) Finite (vi) Infinite (vii) Finite (viii) Infinite.

7. (i) Equal (ii) Equal (iii) Unequal (iv) Unequal

9. (i) Empty (ii) Non-empty (0 is an element of the set)

(iii) Empty (iv) Empty (v) Non-empty

(vi) Non-empty (3 is an element) (vii) Empty (viii) Empty.

EXERCISE 1.2

1. (i) or (ii) and (iii) $x \notin \mathrm{A}$ and $x \notin \mathrm{B}$ (iv) $x \notin \mathrm{B}$ (v) $x \notin \mathrm{A}$ and $x \notin \mathrm{B}$.

2. (i) $\{a, b, c, d, e, x, y, z\}$ (ii) $\{b, c, p, x, y, z\}$ (iii) $\{a, b, c, d, e, p, x, z\}$

(iv) ϕ (v) $\{x, z\}$ (vi) $\{b, c\}$(vii) $\{a, b, c, d, e\}$ (viii) $\{y\}$ (ix) $\{p, x, z\}$

(x) $\{x, y, z\}$ (xi) $\{b, c, p\}$ (xiii) both are $\{a, b, c, d, e, p, x, y, z\}$

(xiv) $\{b, c\}$ (xv) $\{a, b, c, d, e, x, z\}$

3. (i) $\{3, 4, 6, 8, 9\}$ (ii) $\{1, 3, 4, 6, 7, 8\}$ (iii) $\{1, 2, 3, 4, 6\}$ (iv) $\{9\}$

(v) $\{2\}$ (vi) $\{3, 4, 6\}$ (vii) $\{3, 4, 6, 8\}$ (viii) $\{1, 2, 3, 4, 6, 8, 9\}$

(ix) $\{1, 2, 3, 4, 6, 7, 8\}$ (x) $\{9\}$ (xi) $\{2, 3, 4, 5, 6, 7, 8, 9\}$

(xii) $\{1, 2, 7\}$ (xiii) $\{1, 3, 4, 6, 8, 9\}$ (xiv) $\{1, 3, 4, 6, 8, 9\}$

4. (i) If $\mathrm{A} = \{1, 2, 3\}$ and $\mathrm{B} = \{5, 6\}$ then $\mathrm{A} - \mathrm{B} = \mathrm{A}$ but $\mathrm{B} \neq \phi$

(ii) If A = {1, 2, 3, 4} and B = {2, 4} then A∪B=A but B≠ϕ (iii) Take as in (i)

(iv) Take as in (ii) and verify (A∪B)–B≠A

(v) If A = {1, 2, 3} and B = {1, 2, 3, 4, 5} the A–B = ϕ but A≠B

(vi) If A={1, 2, 3} and B={2, 3, 4, 5} and C={1, 4, 5} and A∪B=A∪C but B≠C

(vii) Take A = {1, 2, 3} ; B = {2, 3, 4, 5} and C = {2, 3, 8, 6}

EXERCISE 1.3

1. 28 2. 4 3. 288 4. 34
5. 23 [Hints : Take A as the set of women and children who left the camp and B as the set of woment who reamined behind. then
n(A∪B) = 128 – 25, n(B) = 59 – 14, n(A∩B) = 0 etc.]
6. 10 [Hints : A = set of blue pencils with earaser B = set of black pencils with eraser. C = set of black pencils without eraser. n(B∪C) = 25 – 14 = n(B) + n(C)
n(A∪B) = 16=n(A)+n(B) etc.]
7. 30 8. (i) 20% (ii) 10% 9. 370, 230.

EXERCISE 1.4

1. (i) A×A = {(1, 1), (1, 2), (2, 1), (2, 2)}
A×B = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)}
B×A = {(2, 1), (3, 1), (4, 1), (2, 2) (3, 2), (4, 2)}
B×B = {2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

(ii) A×A = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)}
A×B = {(a, c), (a, f), (g, a), (b, c), (b, e), (b, f), (b, g), (c, e), (c, f), (c, g)}
B×A = {(e, a), (f, a), (g, a), (e, b), (f, b), (g, b), (e, c), (f, c), (g, c)}
B×B = {(e, e), (e, f), (e, g), (f, e), (f, f), (f, g), (g, e), (g, f), (g, g)}

(iii) A×A = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, e), (c, d), (d, a), (d, b), (d, c), (d, d)}
A×B = {(a, 2), (a, 5), (b, 2), (b, 5), (c, 2), (c, 5), (d, 2), (d, 5)}
B×A = {(2, a), (5, a), (2, b), (5, b), (2, c), (5, c), (2, d), (5, d)}
B×B = {(2, 2), (2, 5), (5, 2), (5, 5)}

2. (1, 3), (3, 1), (3, 2)
{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3)}

3. $(a, b), (a, d), (b, a)$
5. (1, 10), (2, 8), (3, 6), (4, 4), (5, 2)
   Domain = {1, 2, 3, 4, 5}, Range = {2, 4, 6, 8 10}
6. Yes, the identity relation in a set A is symmetric as well as the anti-symmetric.
15. R is reflexive and symmetric but not transitive and hence not an equivalence relation.

EXERCISE 1.5

1. (i) Domain = {1, 2, 3, 4}, Range = $\{a, b, c\}$
   (ii) Not a function.
   (iii) Domain = {1, 2, 3, 4, 5, 6}, Range = {1, 2, 3}
   (iv) Not a function.
2. f maps every rational number to 1 and every irrational number to 0.
   $f(1) = 1, f(-2) = 1, f\left(\frac{1}{3}\right) = 1, \quad f\left(\sqrt{2}\right) = 0, \quad f\left(2+\sqrt{3}\right) = 0.$
3. Domain = A = {1, 2, 3, 4} = co-domain, Range = {2, 4}
   (i) No (ii)
4. $f_1 = \{(1, a), (2, a)\}, f_2 = \{(1, b), (2, b)\}, f_4 = \{(1, a), (2, b)\}, f_4 = \{(1, b), (2, a)\}$
   f_3 and f_4 are bijective.
5. $f(x) = 2x-1, x \in A$. f is injective by neither surjective nor bijective.
7. Range of $\phi = \{(2n-1) : n \in Z\}$ = set of all odd integers.
8. ϕ-image of $5 = \phi(5) = 13$
   Pre-image of 5 = 1 (since $f(1) = 2+3=5$)
9. (i) one-one (ii) many-one (iii) one-one (iv) one-one (v) one-one.
10. (i) into (ii) onto (iii) into (iv) onto (v) onto.
11. Range = $\{1^2, 3^2, 5^2 \ldots\ldots\ldots\} \cup \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots\ldots\ldots\right\}$

    $= \{(2r-1)^2 : r \in N\} \cup \left\{\frac{1}{2r} : r \in N\right\}$

    f is one-one

    $f(3) = 3^2 = 9, f(4) = \frac{1}{4}, f(5) = 25$ and $f(16) = \frac{1}{16}$

Chapter–2

RATIO AND PROPORTION

2.1 Ratio :

In mathematics, we often want to compare two quantities. One means of doing this is by what we called ratio. Suppose the weight of a boy is 30 kg and that of a man is 60 kg. Since

$$\frac{30\,\text{kg}}{60\,\text{kg}} = \frac{1}{2},$$

we say that the weight of the boy is half that of the man. Here, the abstract number $\frac{1}{2}$ is the ratio of the weight of the boy to the weight of the man.

Definition : The ratio of one quantity to another of the same kind is defined to be the abstract number (integral or fractional) which expresses what multiples, part or parts the former is of the latter.

Thus, the ratio of two quantities of the same kind is determined by the fraction whose numerator is the measure of the first quantity and whose denominator is the measure of the second quantity, both the quantities being expressed in terms of the same unit.

The ratio of any number a to any number b is expressed by the notation $a : b$ (read "a is to b") which is equivalent to the fractional form $\frac{a}{b}$. The numbers a and b are called terms of the ratio $a : b$. The first number a is called the *antecedent* (or the first term) and the second number b, the consequent (or the second term) of the ratio $a : b$.

Note : The value of a ratio does not depend upon the nature of the quantities involved. It is an abstract number and has no unit.

The *inverse ratio* of a to b is the ratio $b : a$.

The ratio of the product of the antecedents of any number of ratios to the product of their consequents is called the ratio *compounded* of the given ratios. Thus, the ratio compounded of the ratios $a : b$ and $c : d$ is $ac : bd$.

When a given ratio is compounded with itself, the resulting ratio is called *duplicate* of the given ratio. Thus, $a^2 : b^2$ is the duplicate ratio of $a : b$. Similarly, $a^3 : b^3$ is called the *triplicate ratio* of $a : b$; $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called *sub-duplicate* ratio and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, the *sub-triplicate* ratio of $a : b$.

Example 1 : If $2a+3b : 3a+5b = 13 : 21$, find $a : b$.

Solution : We have,

$$\frac{13}{21} = \frac{2a+3b}{3a+5b} = \frac{2\cdot\frac{a}{b}+3}{3\cdot\frac{a}{b}+5} \quad \text{(dividing both numerator and denominator by } b\text{)}$$

$$\Rightarrow \quad 13\left(3\cdot\tfrac{a}{b}+5\right) = 21\left(2\cdot\tfrac{a}{b}+3\right)$$

$$\Rightarrow \quad 39\cdot\tfrac{a}{b} + 65 = 42\cdot\tfrac{a}{b} + 63$$

$$\Rightarrow \quad 3\cdot\frac{a}{b} = 2$$

$$\Rightarrow \quad \frac{a}{b} = \frac{2}{3}$$

i.e. $a : b = 2 : 3$.

Example 2 : Two numbers are in the ratio $3 : 4$. If 11 be added to each, the sums are in the ratio $4 : 5$. Find the numbers.

Solution : Let the numbers in the ratio $3 : 4$ be $3x$ and $4x$.

By the given condition,

$$3x+11 : 4x+11 = 4 : 5$$

$$\Rightarrow \quad \frac{3x+11}{4x+11} = \frac{4}{5}$$

$$\Rightarrow \quad 15x+55 = 16x+44$$

$$\Rightarrow \quad x = 11$$

Therefore, the numbers are 3×11 and 4×11 i.e. 33 and 44.

Example 3 : If a, b are positive, show that the ratio $a+b : a^2+b^2$ is greater than the ratio $a-b : a^2-b^2$.

Solution : We have,

$$\frac{a+b}{a^2+b^2} - \frac{a-b}{a^2-b^2} = \frac{a+b}{a^2+b^2} - \frac{a-b}{(a+b)(a-b)}$$

$$= \frac{a+b}{a^2+b^2} - \frac{1}{a+b}$$

$$= \frac{(a+b)^2-(a^2+b^2)}{(a^2+b^2)(a+b)}$$

$$= \frac{2ab}{(a^2+b^2)(a+b)} > 0 \quad (\because a,\ b \text{ are positive})$$

$$\therefore \quad \frac{a+b}{a^2+b^2} > \frac{a-b}{a^2-b^2}$$

2.2. Proportion :

Four quantities are said to be in *proportion* if the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus, a, b, c, and d are in proportion if $a : b = c : d$. Here a and d are called the *extremes* and b and c, the *means*. The last term d is also called the *fouth proportional* to a, b, c.

It is easy to see that *the product of the extremes in a proportion is equal to the product of the means*.

Three quantities a, b, c are said to be in *continued proportion* if $a : b = b : c$. Here, b is called the *mean proportional* between a and c and c is called the *third proportional* to a and b.

If a, b, c are in continued proportion, then

$$\frac{a}{b} = \frac{b}{c}$$

$$\Rightarrow \quad b^2 = ac$$

$$\Rightarrow \quad b = \sqrt{ac}$$

Example 1. Find the mean proportional between 3 and 27.

Solution : Let x be the mean proportional between 3 and 27.

Then 3, x, 27 are in continued proportion

$$\text{i.e.} \quad \frac{3}{x} = \frac{x}{27}$$

$$\Rightarrow \quad x^2 = 3 \times 27$$

$$\Rightarrow \quad x = \sqrt{3 \times 27} \quad = 9.$$

The mean proportional is 9.

Example 2 : Find the third proportional to 6 and 15.

Solution : Let x be the third proportional to 6 and 15.

Then, 6, 15, x are in continued proportion

$$\text{i.e.} \quad \frac{6}{15} = \frac{15}{x}$$

$$\Rightarrow \quad 6x = 15 \times 15$$

$$\Rightarrow \quad x = \frac{15 \times 15}{6} \quad = 37\frac{1}{2}$$

The third proportional is $37\frac{1}{2}$.

Example 3 : Find the fourth proportional to 12, 16, 48.

Solution : Let x be the fouth proportional to 12, 16, 48.

Then, 12, 16, 48, x are in proportion

i.e. $\frac{12}{16}=\frac{48}{x}$

$\Rightarrow \quad 12x = 16\times48$

$\Rightarrow \quad x = \frac{16\times48}{12} = 64$

The fourth proportional is 64.

2.3. Rules of proportion :

I. If $a : b = c : d$, then $b : a = d : c$.

For, $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \quad 1\div\frac{a}{b}=1\div\frac{c}{d}$

$\Rightarrow \quad \frac{b}{a}=\frac{d}{c}$

i.e. $b : a = d : c$

This operation is known as *Invertendo*.

II. If $a : b = c : d$, then $a : c = b : d$.

For, $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \quad \frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{b}{c}$ (Multiplying both sides by $\frac{b}{c}$)

$\Rightarrow \quad \frac{a}{c}=\frac{b}{d}$

i.e. $a : c = b : d$

This operation is known as *Alternando*.

III. If $a : b = c : d$, then $a+b : b = c+d : d$.

For, $\frac{a}{b}=\frac{c}{d}$

$\Rightarrow \quad \frac{a}{b}+1=\frac{c}{d}+1$

$\Rightarrow \quad \frac{a+b}{b}=\frac{c+d}{d}$

i.e. $a + b : b = c + d : d$

This operation is known as *Componendo*.

IV. If $a : b = c : d$, then $a-b : b = c-d : d$.

For, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{b} - 1 = \frac{c}{d} - 1$

$\Rightarrow \quad \frac{a-b}{b} = \frac{c-d}{d}$

i.e. $a-b : b = c-d : d$.

This operation is known as *Dividendo*.

2.4. Some deductions from the rules of proportion :

(A) If $a : b = c : d$, then $a : a-b = c : c-d$.

For, $\frac{a}{b} = \frac{c}{d}$ .... (i)

$\Rightarrow \quad \frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

$\Rightarrow \quad \frac{b}{a-b} = \frac{d}{c-d}$ .... (ii) (Invertendo)

From (i) and (ii) $a : a-b = c : c-d$.

This operation is known as *Convertendo*.

(B) If $a : b = c : d$, then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

For, $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{c} = \frac{b}{d}$ (Alternando)

$\Rightarrow \quad \frac{a+c}{c} = \frac{b+d}{d}$ (Componendo)

$\Rightarrow \quad \frac{a+c}{b+d} = \frac{c}{d}$ (Alternando)

$\Rightarrow \quad \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

Similarly, it can be proved that, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots\ldots\ldots$, then each of the ratios $= \frac{a+c+e+\ldots\ldots\ldots\ldots}{b+d+f+\ldots\ldots\ldots\ldots}$

This operation is known as *Addendo*.

(C) If $a : b = c : d$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

Since $\dfrac{a}{b} = \dfrac{c}{d}$, therefore by componendo

$$\frac{a+b}{b} = \frac{c+d}{d} \quad \text{.................................. (i)}$$

and by dividendo

$$\frac{a-b}{b} = \frac{c-d}{d} \quad \text{.................................. (ii)}$$

Dividing (i) by (ii) we obtain,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

This operation is known as *Componendo and Dividendo*.

2.5. Theorem :

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then each of the ratios $= \left(\dfrac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n} \right)^{\frac{1}{n}}$, where p, q, r, n are any quantities whatsoever.

Proof : Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$.

Then $a = bk$, $c = dk$, $c = fk$.

$$\therefore \quad pa^n + qc^n + re^n = p\,(bk)^n + q\,(dk)^n + r\,(fk)^n$$

$$= \left(pb^n + qd^n + rf^n\right) k^n$$

$$\Rightarrow \quad k^n = \frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}$$

$$\Rightarrow \quad k = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n} \right)^{\frac{1}{n}}$$

This proves the theorem.

Example 1 : If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{9}{7}$, find the values of (i) $\dfrac{a+b}{a-b}$ (ii) $\dfrac{a+c}{b+d}$ (iii) $\dfrac{ad}{bc}$ (iv) $\dfrac{a-c}{b-d}$

Solution : (i) Since $\dfrac{a}{b} = \dfrac{9}{7}$

$$\therefore \quad \frac{a+b}{a-b} = \frac{9+7}{9-7} \quad \text{(Componendo and dividendo)}$$

$$= \frac{16}{2} = 8$$

(ii) Since $\frac{a}{b}=\frac{c}{d}=\frac{9}{7}$

$\therefore \quad \frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}=\frac{9}{7}$ (Addendo)

i.e. $\frac{a+c}{b+d}=\frac{9}{7}$

(iii) Since $\frac{a}{b}=\frac{c}{d}$

$\therefore \ ad = bc$ (product of extremes=product of means)

$\Rightarrow \frac{ad}{bc}=1$

(iv) Since $\frac{a}{b}=\frac{c}{d}$

$\therefore \ \frac{a}{b}=\frac{(-1)\,c}{(-1)\,d}$

$\Rightarrow \frac{a}{b}=\frac{-c}{-d}=\frac{a-c}{b-d}$ (Addendo)

$\Rightarrow \frac{9}{7}=\frac{a-c}{b-d}$ $\left(\frac{a}{b}=\frac{9}{7}\right)$

$\Rightarrow \frac{a-c}{b-d}=\frac{9}{7}$

Example 2 : If $x=\frac{4\sqrt{6}}{\sqrt{2}+\sqrt{3}}$, find the value of $\frac{x+2\sqrt{2}}{x-2\sqrt{2}}+\frac{x+2\sqrt{3}}{x-2\sqrt{3}}$

Solution : Since $x=\frac{4\sqrt{b}}{\sqrt{2}+\sqrt{3}}=\frac{2\sqrt{2}.2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

therefore, $\frac{x}{2\sqrt{2}}=\frac{2\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ .... (i)

and, $\frac{x}{2\sqrt{3}}=\frac{2\sqrt{2}}{\sqrt{2}+\sqrt{3}}$ .... (ii)

From (i), $\frac{x+2\sqrt{2}}{x-2\sqrt{2}}=\frac{2\sqrt{3}+\sqrt{2}+\sqrt{3}}{2\sqrt{3}-\sqrt{2}-\sqrt{3}}$ (Componendo and dividendo)

$=\frac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

From (ii), $\dfrac{x+2\sqrt{3}}{x-2\sqrt{3}}=\dfrac{2\sqrt{2}+\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{2}-\sqrt{3}}$ (Componendo and dividendo)

$$=\frac{3\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$$

Now, $\dfrac{x+2\sqrt{2}}{x-2\sqrt{2}}+\dfrac{x+2\sqrt{3}}{x-2\sqrt{3}}=\dfrac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\dfrac{3\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$

$$=\frac{3\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{3\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$$

$$=\frac{3\sqrt{3}+\sqrt{2}-3\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}=\frac{2\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)}=2$$

Example 3 : If $(a+3b+2c+6d)(a-3b-2c+6d)=(a-3b+2c-6d)(a+3b-2c-6d)$, prove that $a:b=c:d$.

Solution : Since, $(a+3b+2c+6d)(a-3b-2c+6d)$

$$=(a-3b+2c-6d)(a+3b-2c-6d)$$

therefore, $\dfrac{(a+3b)+(2c+6d)}{(a+3b)-(2c+6d)}=\dfrac{(a-3b)+(2c-6d)}{(a-3b)-(2c-6d)}$ (why ?)

$\Rightarrow \dfrac{a+3b}{2c+6d}=\dfrac{a-3b}{2c-6d}$ (Componendo and dividendo)

$\Rightarrow \dfrac{a+3b}{c+3d}=\dfrac{a-3b}{c-3d}$

$\Rightarrow \dfrac{a+3b}{a-3b}=\dfrac{c+3d}{c-3d}$ (Alternando)

$\Rightarrow \dfrac{a}{3b}=\dfrac{c}{3d}$ (why ?)

$\Rightarrow \dfrac{a}{b}=\dfrac{c}{d}$

Hence, $a:b=c:d$.

Example 4 : If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\left(\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}$

Solution : Since $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$

therefore, $\frac{a^2}{b^2}=\frac{c^2}{d^2}=\frac{e^2}{f^2}$

$$= \frac{a^2+c^2+e^2}{b^2+d^2+f^2} \qquad \text{(Addendo)}$$

Therefore $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}$ (Taking Square roots)

Second Method :

Supose $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$

then, $a=bk$; $c=dk$; $e=fk$.

$$\therefore \sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}=\sqrt{\frac{b^2k^2+d^2k^2+f^2k^2}{b^2+d^2+f^2}}=\sqrt{\frac{k^2(b^2+d^2+f^2)}{b^2+d^2+f^2}}=\sqrt{k^2}=k$$

Hence, $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}}=k$

Example 5 : If $\frac{a}{3}=\frac{b}{4}=\frac{c}{7}$, show that $\frac{a+b+c}{c}=2$

Solution : Since $\frac{a}{3}=\frac{b}{4}=\frac{c}{7}$

therefore, each ratio $=\frac{a+b+c}{3+4+7}$

$\Rightarrow \frac{c}{7}=\frac{a+b+c}{14}$

$\Rightarrow \frac{a+b+c}{c}=\frac{14}{7}=2$ (Alternando)

Hence, $\frac{a+b+c}{c}=2$

EXERCISE 2.1

1. If $a:b=2:3$, find the value of $3a+5b:7a+2b$.
2. If $2a+5b:3a+2b=26:27$, find $a:b$.
3. Two numbers are in the ratio 8 : 9 and their sum is 204. Find the numbers.
4. Two numbers are in the ratio 7 : 5 and their difference is 60. Find the numbers.
5. Two numbers are in the ratio 3 : 5 and if 2 be added to each, the sums are in the ratio 5 : 8. Find the numbers.
6. Find the value of x for which the ratio $17-x:13-x$ is equal to 3.

7. What number must be added to each terms of the ratio 23 : 27 so that it may become equal to 10 : 11 ?

8. Find the third proportional to :

 (i) 4, 6, (ii) 16, 12 (iii) 24, 36 (iv) 125,

9. (a) Find the mean proportional between :

 (i) 5, 125 (ii) 3, 27 (iii $\frac{1}{12}$, 48

 (b) If $\frac{1}{3}$, x, 27 are in continued proportion, find x.

10. Find the fourth proportional to :

 (i) 3, 4, 6 (ii) 14, 24, 35 (i $\frac{1}{\sqrt{3}}$, $\sqrt{3}$, 13

11. If $a : b = c : d$, prove the following :

 (i) $a : b = a + c : b + d = a - c : b - d$

 (ii) $a^2 : b^2 = a^2 + c^2 : b^2 + d^2$

 (iii) $a^2 + c^2 : b^2 + d^2 = ac : db$

 (iv) $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}=\sqrt{3a^2+5b^2}:\sqrt{3c^2+5d^2}$

 (v) $ma+nc:mb+nd=(a^2+c^2)^{\frac{1}{2}}:(b^2+d^2)^{\frac{1}{2}}$

12. If $a : b = b : c$, show that

 (i) $a^2+ab+b^2:b^2+bc+c^2=a:c$ (ii) $a^2b^2c^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=a^3+b^3+c^3$

 (iii) $a^3+b^3:b^3+c^3=a^3:b^3$

13. If a, b, c, d, e are in continued proportion, prove that

 (i) $a+d:b+e=a+b+c+d:b+c+d+e$

 (ii) $a:e=a^4:b^4$

14. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that

 (i) $\frac{x^2-yz}{a^2-bc}=\frac{y^2-zx}{b^2-ca}=\frac{z^2-xy}{c^2-ab}$

 (ii) $\frac{a^3+b^3+c^3}{x^3+y^3+z^3}=\frac{abc}{xyz}$

 (iii) $\frac{x^3}{a^3}+\frac{y^3}{b^3}+\frac{z^3}{c^3}=\frac{3xyz}{abc}$

15. If $x=\dfrac{2ab}{a+b}$, find the value of $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}$.

16. If $x=\dfrac{a+b}{a-b}$ and $y=\dfrac{a-b}{a+b}$, find the value of $\dfrac{x-y}{x+y}$.

17. If $x=\dfrac{2\sqrt{10}}{\sqrt{2}+\sqrt{5}}$, find the value of $\dfrac{x+\sqrt{2}}{x-\sqrt{2}}+\dfrac{x+\sqrt{5}}{x-\sqrt{5}}$.

18. Prove that $3bx^2-4ax+3b=0$ if $x=\dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$.

19. If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, show that

(i) $\dfrac{a^3+b^3+c^3}{b^3+c^3+d^3}=\dfrac{a}{d}$.

(ii) $\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}:\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a:b$.

(iii) $(b+c)(b+d)=(c+a)(c+d)$.

(iv) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$.

20. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, show that $(b-c)x+(c-a)y+(a-b)z=0$

21. If $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{b}{x}$, show that $x^2=2ab-b^2$.

22. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{(a^2+c^2)^2}{(b^2+d^2)^2}=\dfrac{a^4+c^4}{b^4+d^4}$.

ANSWERS

1. 21 : 20 2. 3 : 4 3. 96 and 108 4. 210 and 150 5. 18 and 30

6. 11 7.11

8. (i) 9 (ii) 9 (iii) 54 (iv) 45

9. (a) (i) 25 (ii) 9 (iii) 2 9.(b) 3

10. (i) 8 (ii) 60 (iii) 39

15. 2 16. $\dfrac{2ab}{a^2+b^2}$ 17. 2

Chapter–3

MATHEMATICAL LOGIC

3.1 Introduction

In mathematics, we prove many theorems. The proofs are based on sound reasonings or arguments. The main object of logic is to provide a set of rules by which one can determine whether any particular reasoning or argument is valid or not. In the present chapter, we shall discuss some of the logical terminologies and define some of its concepts. One can decide about the validity of an argument by following mechanically the most fundamental rules of logic. In what follows, we shall study about statement calculus, structure of simple and compound statements, logical connectives and negations of different types of compound statements.

3.2 Statements

In logic, we are concerned with only declarative or assertive sentences called statements.

Definition : A statement is an assertive sentence which is either true or false, but not both simultaneously.

We shall use letters such as p, q, r etc. to denote statements.

Example : Consider the following sentences :

(I) 5 is a prime number.

(II) The sum of two even integers is an even integer.

(III) The earth revolves round the sun.

(IV) 3 is greater than 5.

(V) Every rectangle is a square.

(VI) The square of every real number is negative.

All these six sentences are statements, the first three being true and last three being false.

Again, consider the following sentences :

(a) Close the door.

(b) Will you go to school ?

(c) May God bless you.

These sentences are not logical statements for it is meaningless to declare any of them to be true or false. In fact, imperative, interrogative and exclamatory sentences are not statements. Also, assertive sentences are not always statements. For example, consider the sentence "What I am saying is false". If we assign the value 'true', this sentence asserts that what I am saying, in particular the sentence itself is false. On the other hand if we assign to it value 'false', then the sentence asserts what I am saying is true, that is the sentence itself is true. This example illustrates what we call a 'semantic paradox'. Thus, every assertive sentence need not be a statement. In the following, an interesting example is given, of a situation where a semantic paradox saves life :

A missionary, on his way to spread religion, was caught by a group of cannibals. The chief of the cannibals said, "We will kill you either by roasting or by cutting into pieces. You make a statement ; if what you say is true, you will be roasted and if what you say is false, you will be cut into pieces". Then said the missionary, "I will be cut into pieces."

If what the missionary says is true, he must be roasted, not to be cut into pieces so that his statement becomes false. On the other hand, if what he says is false, he should be cut into pieces so that his statement becomes true. The chief, having thus faced with semantic paradox could not kill the missionary either by roasting or by cutting into pieces and so set him free.

3.3 Truth Value of a Statement

The truth or falsity of a statement is called its *truth value*. Every statement must be either true or false and cannot be both true and false at the same time. If a statement is true, we say that its truth value is 'truth' symbolically denoted by 'T' and if it is false, we say that its truth value is 'falsity' denoted by 'F'. Thus, any statement has one and only one of T and F as its truth value.

3.4 Simple and Compound Statements

A statement which does not contain any other statement as its part is called a *simple (or an atomic or a primary)* statement. In other words, a statement is said to be simple if it cannot be broken up into two or more statements.

The following are examples of simple statements :

(i) A triangle has three sides.

(ii) The sum of the interior angles of a triangle is two right angles.

(iii) The sky is blue.

A statement formed by combining two or more simple statements is called a *compound* (or *molecular*) statement. Thus, a compound statement is one which is made up of two or more simple statements.

Examples :–

(i) The statement "Roses are red and sky is blue" is a compound statement which is a combination of two simple statements "Roses are red" and "Sky is blue".

(ii) The statement "If there is a will, then there is a way" is a compound statement made up of two simple statements "There is a will" and "There is a way".

The simple statements, which on combining, form a compound statement, are called sub-statements or component statements of the compound statement.

3.5 Logical connectives

There are many ways of combining simple statements to form compound statements. The words or phrases which combine simple statements to form compound statements are known as *sentencial* or *logical connectives.* Main logical connectives are 'And', 'Or', 'Not', 'If .... then' and 'If and only if'. We shall study these connectives and deduce a set of rules how to assign truth values to statements involving them. These rules enable us to do calculations by using statements as objects and find truth values of complicated statements.

3.6 Conjunction

When two statements are joined by the connective 'and', the compound statement so formed is called the *conjunction* of the two statements. Thus, if p and q are two statements, then their conjunction is 'p and q' and it is denoted by the symbol '$p \wedge q$'.

The statement '$p \wedge q$' is true when both p and q are true, otherwise it is false. The truth values of $p \wedge q$ for different combinations of the truth values of the component statements p and q are given in the following table, called "*truth functional rules*" or "truth table" for $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for Conjunction

Example 1. Form the conjunction of

p : 2+3=5,

q : 5 is a positive integer.

What is its truth-value ?

Solution : 2+3=5 and 5 is a positive integer.
Since both p and q have the truth-value T, therefore $p \wedge q$ i.e. the conjunction has also the truth-value T.

Example 2. Represent the following conjunction in symbolic form :
Ibohal is rich and healthy.

Solution : Let us first paraphrase the statement as
"Ibohal is rich and Ibohal is healthy".
If we denote
p : Ibohal is rich,
q : Ibohal is healthy,

then the given conjunction in symbolic form becomes $p \wedge q$.

From the above table, we see the truth values of $p \wedge q$ must be the same as those of $q \wedge p$. In other words, the two statements $p \wedge q$ and $q \wedge p$ are treated as identical in logic. Now consider the statement 'He entered the hall and sat on a chair'. This statement does not have the same meaning as 'He sat on a chair and entered the hall'. In this example, the word "and" has been used in the sense of "and then" and therefore cannot be represented by the symbol $\wedge$ in logical sense.

3.7 Disjunction

When two statements are joined by the connective 'or', the compound statement so formed is called *disjunction of the two statements*. The disjunction of the statements p and q is denoted by "$p \vee q$" and read as "p or q". The disjunction $p \vee q$ is always true except when both p and q are false and its truth values are as shown below :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table for Disjunction

Consider the statement 'The driver is drunk or the brakes are defective'. Here the word 'or' is used in inclusive sense, for though one of the statements

p : The driver is drunk,

q : The brakes are defective,

is asserted, the assertion of both is not excluded and it may be a case that the driver is drunk and the brakes are also defective. On the other hand, in the statement 'I shall watch the cricket match on television or go to the stadium' the connective 'or' is used in the exclusive sense for one or other possibility exists but not both.

In logic, the connective $\vee$ is used as 'inclusive or'. Thus $p \vee q$ usually means 'either p or q or both'. Again let us consider the statement "Thirty or forty students were present today'. Here the word 'or' is not used as a connective, but to indicate an approximate number of students and so it cannot be represented by '$\vee$' in the logical sense.

3.8 Negation

Negation is the denial of the assertion made in the statement. The negation of a statement is usually formed by introducing the word 'not' at a proper place in the statement or by prefixing the statement with one of the phrases 'It is not the case that' and 'It is not true that'.

The negation of a statement p is denoted by '$\sim p$' and read as "tilde p" or "not p" The truth values of a statement and its negation are always opposite. This is summarized in the following table :

p	$\sim p$
T	F
F	T

Truth Table for Negation

Note : The name of the symbol '$\sim$' is tilde.

If p stands for the statement '$5^3 = 125$', then $\sim p$ is any one of the following :

(i) $5^3 \neq 125$.

(ii) It is not the case that $5^3 = 125$.

(iii) It is not true that $5^3 = 125$.

Sometimes, the negation of a statement cannot be formed only by the introduction of the word 'not'. For example, consider the statement :

p : All parallelograms are rectangles.

Here the statement 'All parallelograms are not rectangles' is not the negation of p for both the statements have logically the same truth value F. Here '$\sim p$' may be stated as any one of the following :

(i) It is not true that all parallelograms are rectangles.

(ii) Some parallelograms are not rectangles.

(iii) There exist parallelograms which are not rectangles.

3.9 Implication or Conditional :

The compound statement formed by conjoining two statements by the connective 'If .... then', is called an *implication* (or a *conditional*).

For example, the statement

"If it rains, then there will be no play" .... (1)

is an implication. If p and q are the statements

p : It rains

q : There will be no play,

then the statement (1) will be denoted by "$p \Rightarrow q$" (read as 'p implies q')

In the conditional '$p \Rightarrow q$' the statement p is called antecedent or condition or hypothesis and q is called consequent or conclusion. The name 'conditional' is given to the compound statement "$p \Rightarrow q$" in the sense that the conclusion q is drawn subject to the condition p.

The truth values of $p \Rightarrow q$ are as tabulated below :

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for Implication

It may be noted that '$p \Rightarrow q$' is false only when p is true and q is false and further that when p is false, '$p \Rightarrow q$' is true irrespective of the truth values of q.

Example 3. Express the statement $p \Rightarrow q$ in words when

p : $5 < 8$

q : The sun rises in the east.

State its truth value.

Solution : If $5 < 8$, then the sun rises in the east.

Since the truth values of both p and q are T, therefore, the truth value of $p \Rightarrow q$ is also T, that is the implication is true.

Example 4. For the statements

p : $5 \in \{1, 2, 3\}$

q : $6 \in \{4, 5, 6\}$

r : $5 + 6 = 11$

express in words the statement $p \wedge q \Rightarrow r$. Explain why this compound statement is true.

Solution : If $5 \in \{1, 2, 3\}$ and $6 \in \{4, 5, 6\}$, then $5 + 6 = 11$.

Since p i.e. $5 \in \{1, 2, 3\}$ is false and q is true, therefore the conjunction $p \wedge q$ is false. Thus, in the implication $p \wedge q \Rightarrow r$, the antecedent $p \wedge q$ is false and so the implication is true irrespective of the truth value of the consequent r (which is also true).

Note : Each of the following expressions is used synonymous with '$p \Rightarrow q$'.

(i) If p then q
(ii) p implies q
(iii) p only if q
(iv) q if p
(v) q is necessary for p
(vi) p is sufficient for q

3.10 Double Implication or Two-way Implication or Biconditional

If p and q are any two statements, the compound statement 'p if and only if q' is called a *double* or *two-way implication* or a *biconditional statement*. The phrase 'if and only if' is often abbreviated as '*iff*' and the statement 'p *iff* q' is denoted as '$p \Leftrightarrow q$'.

The biconditional $p \Leftrightarrow q$ may be verbally asserted in different forms as follows :

1. p implies q and conversely.
2. If p then q and conversely.
3. p implies and is implied by q.
4. p is necessary and sufficient condition for q.
5. p if and only if q.
6. p is equivalent to q.

The statement $p \Leftrightarrow q$ is true only when p and q have identical truth values.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for Biconditional.

It may be noted the $p \Leftrightarrow q$ is precisely the conjunction of $p \Rightarrow q$ and $q \Rightarrow p$, that is, the statement $p \Leftrightarrow q$ has the same meaning as $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Example 5. If

p : 15 is a prime

q : 13 is even,

write the expression for $p \Leftrightarrow q$ and state whether this statement is true or false.

Solution : Here $p \Leftrightarrow q$ is the statement "15 is a prime if and only if 13 is even". As p and q have the same truth value (i.e. F), therefore the biconditional $p \Leftrightarrow q$ is true.

Remarks : Unlike in our everyday language, in logic it is perfectly acceptable to form compound statements from statements which have no kind of relation in usual sense. As for example, the following statements are all acceptable in logic :

(I) $5 > 3$ and today is Sunday.

(II) 2 is prime or the grass is green.

(III) If you pass, then the sky is red.

(IV) If it rains, then $2 + 2 = 4$.

3.11 Open Statement

Some assertive sentences involving one or more unknowns or unspecified terms are such that their truth values depend on the values of the unknowns or unspecified terms and consequently their truth values can be determined only when the values of the unknowns are determined. Such a sentence is called on open statement.

Definition : An open statement is an assertive sentence involving one or more unknowns or unspecified terms and whose truth value depends on these unknowns or unspecified terms.

Examples : (I) $3x - 5 = 13$.

(II) He is ex-principal of the college.

(III) $x + y = 1$.

All the above assertive sentences are open statements. The first one has the truth value T only when the unknown x is the number 6. When x assumes any other number as its value, it is false statement. In the second sentence, the words "He" and "the College" are unspecified terms and the truth value of the sentence depends on these terms. In the last sentence, x and y are the unknowns which evidently assume real numbers as their values and the truth value of the sentence depends on the values these unknowns assume.

3.12 Construction of Truth Tables

To construct a truth table for a given statement, we first study the logical connectives in the statement separately and deduce the truth values of conjunction, disjunction, implication etc. that occur as constituents of the statement and ultimately deduce the truth values of the whole statement associated with different combinations of the truth values of the component atomic statements. The process will be clear from the following examples.

Example 6 : Construct the truth table for the statement $(p \wedge q) \vee \sim p$.

Solution : The statement consists of only two independent atomic components p and q. We consider all possible combinations of the truth values of p and q. These values are entered in the first two columns of the table.

1	2	3	4	5
p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee \sim P$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

The corresponding truth values of $p \wedge q$ (as given by Truth Table for Conjunction) are entered in the third column. The truth values of $\sim p$ (as obtained from truth table for negation) are entered in the fourth column. The given statement is the disjunction of the statements in columns 3 and 4 and so its truth values may be calculated by observing the truth table for disjunction. These are entered in the last column.

Note : In constructing truth table for a given compound statement, we need not explain the steps unless otherwise demanded.

Example 7 : Construct the truth table for $[q \wedge (p \Rightarrow q)] \Rightarrow p$.

Solution :

p	q	$p \Rightarrow q$	$q \wedge (p \Rightarrow q)$	$q \wedge (p \Rightarrow q) \Rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The given statement is an implication in which the statement in the fourth column is the antecedent and that in the first, the consequent, and so its truth values may be calculated by applying truth functional rules (Truth Table) of implication.

3.13. Negations of Compound Statements

(a) The negation of $p \wedge q$ is $\sim p \vee \sim q$

[i.e. $\sim(p \wedge q) \equiv \sim p \vee \sim q$ or $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$]

Proof : Consider the following truth table :

p	q	$\wedge q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

We see that the truth values of $p \wedge q$, as given in the third column are exactly opposite to those of $\sim p \vee \sim q$ given in the last column. Hence, $\sim p \vee \sim q$ is the negation of $p \wedge q$ [i.e. $\sim(p \wedge q \equiv \sim p \vee \sim q$]

(b) The negation of $p \vee q$ is $\sim p \wedge \sim q$

Proof :

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	T	F	T	F
F	T	T	T	F	F
F	F	F	T	T	T

From the above table, we see that truth values of $p \vee q$ (given in the third column) are exactly opposite to those of $\sim p \wedge \sim q$.

(c) The negation of $\sim p$ is p. [i.e. $\sim(\sim p) \equiv p$]

Proof : Follows from the fact that the truth values of p are exactly opposite to those of $\sim p$

(d) The negation of $p \Rightarrow q$ is $p \wedge \sim q$.

Proof : Follows from the following table.

p	q	$p \Rightarrow q$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	F

(e) The negation of $p \Leftrightarrow q$ is $p \Leftrightarrow \sim q$ or $\sim p \Leftrightarrow q$

Proof : Follows from the table :

p	q	$p \Leftrightarrow q$	$\sim p$	$\sim q$	$p \Leftrightarrow \sim q$	$\sim p \Leftrightarrow q$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	F	F

Example 8 : Write the negation of each of the following statements :

(1) Chaoba is rich and happy.

(2) Chaoba is rich or happy.

(3) If one is rich, then one is happy.

(4) One is happy if and only if one is rich.

Justify the answer in each case.

Solution : (1) Chaoba is poor or unhappy.

(2) Chaoba is poor and unhappy.

(3) One is rich but unhappy.

(4) One is happy if and only if one is poor.

(or, One is unhappy if and only if one is rich)

If p : Chaoba is rich

q : Chaoba is happy,

then statements (1) and (2) are $p \wedge q$ and $p \vee q$ respectively. The negations of these compound statement are $\sim p \vee \sim q$ and $\sim p \wedge \sim q$ respectively.

But $\sim p$: Chaoba is poor.

$\sim q$: Chaoba is unhappy.

so that $\sim p \vee \sim q$: Chaoba is poor or unhappy.

and $\sim p \wedge \sim q$: Chaoba is poor and unhappy.

Again, when

p : One is rich

q : One is happy

the given statements (3) and (4) are $p \Rightarrow q$ and $q \Leftrightarrow p$ respectively, so that their negations are $p \wedge \sim q$ and $q \Leftrightarrow \sim p$ respectively. The expressions for these negations are given above.

3.14 Converse of an Implication

The converse of a given implication is a new implication formed by interchanging the antecedent and the consequent of the given implication. Thus, the converse of $p \Rightarrow q$ is $q \Rightarrow p$. There is no relation between the truth values of an implication and those of its converse in general. Consider the statement

'$x = 4 \Rightarrow x^2 = 16$'

This statement is evidently true, however the converse i.e. the statement

$$`x^2 = 16 \Rightarrow x = 4'$$

is false for even if the square of a number is 16, the number need not be 4 as it may be -4. Again, consider the statement

$$`3x + 5 = 8 \Rightarrow x = 1'$$

Its converse is '$x = 1 \Rightarrow 3x + 5 = 8$'. Here both the statement and its converse are true.

3.15 Tautology and Contradiction

Definition : A compound statement which is always true irrespective of the truth values of its component parts is called a *tautology*.

Definition : A compound statement which is always false irrespective of the truth values of its component parts is called a *contradiction*.

***Examples* :** The statement $(p \Rightarrow q) \Leftrightarrow \sim p \vee q$ is a tautology and the statement $p \wedge \sim p$ is a contradiction. These are verified by the following truth tables :

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \Rightarrow q) \Leftrightarrow \sim p \vee q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

***Example 9* :** Prove that $(p \vee q) \wedge \sim q \Rightarrow p$ is a tautology.

Solution :

①	②	③=①∨②	④ = ~②	⑤=③∧④	⑥=⑤ ⇒ ①
p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge \sim q$	$(p \vee q) \wedge \sim q \Rightarrow p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

The truth values of the statement $(p \vee q) \wedge \sim q \Rightarrow p$ as given in the last column of the above table are all T. Hence the statement is a tautology.

EXERCISE 3.1

1. Which of the following are statements ?
   (a) The square of an integer is an even integer.
   (b) Do you read at night ?
   (c) Come here, Tomba.
   (d) If it rains, then grass grows.
   (e) 13 is a composite number.
   (f) A triangle has four sides.
2. Using the statements
   p : Chaoba is a good teacher.
   q : Chaoba is a scholar.
   Write the following in symbolic form :
   (i) Chaoba is not a good teacher but a scholar.
   (ii) Chaoba is a good teacher but not a scholar.
   (iii) Chaoba is neither a good teacher nor a scholar.
   (iv) Chaoba is a good teacher or he is a scholar and a bad teacher.
3. Given the truth values of p, q and r to be T, F and T respectively, find the truth value of :
   (i) $(p \vee q) \wedge (q \vee r)$
   (ii) $(p \Rightarrow q) \Rightarrow (p \wedge \sim q)$
   (iii) $(p \Rightarrow q) \wedge (q \Rightarrow r)$
   (iv) $(q \wedge r) \Rightarrow p$
   (v) $q \vee (r \Rightarrow p)$
4. Construct truth table for the following statements :
   (i) $(p \wedge q) \vee \sim r$ (ii) $(p \Leftrightarrow q) \wedge (\sim r \Rightarrow p)$
   (iii) $(p \vee \sim q) \wedge r$ (iv) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
5. Write the negations of the following :
   (a) 5 is a rational number.
   (b) 3 is not a prime.
   (c) All integers are rational numbers.
   (d) There are natural numbers which are not integers.

(e) A triangle has four sides.

(f) Man is mortal.

(g) If water is cold, then fire is hot.

(h) Kumar and Kanta are intelligent.

(i) Some students never read.

(j) Every student is honest.

(k) An integer is either positive or negative.

(l) If there is a will, then there is a way.

6. Find the negation of

   (i) $(p \vee q) \wedge r$ (ii) $p \wedge q \Rightarrow r$ (iii) $p \wedge (q \Rightarrow r)$

   Verify your result by constructing truth tables.

7. Examine whether the following statements are tautologies or contradictions :

   (a) $p \Rightarrow p \vee q$

   (b) $p \wedge q \Rightarrow q$

   (c) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

   (d) $(p \wedge q) \wedge \sim(p \vee q)$

   (e) $(p \vee q) \wedge r \Rightarrow (q \vee r)$

   (f) $(p \wedge \sim q) \wedge q \Leftrightarrow p \wedge q$

   (g) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

   (h) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

   (i) $(p \Leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

8. The negation of the statement "3>5 and 5–3=2" is

   A. $3 \not> 5$ and $5 - 3 \neq 2$

   B. $3 > 5$ and $5 - 3 \neq 2$

   C. $3 \not> 5$ or $5 - 3 \neq 2$

   D. $3 \not> 5$ and $5 - 3 = 2$.

9. The negation of 'If it rains, then grass is green' is

   A. If it rains, then grass is not green.

   B. If it does not rain, then grass is green.

   C. It does not rain but grass is green.

   D. It rains but grass is not green.

10. The converse of the statement 'If a whole number is even, then it is divisible by 2' is

   A. An even whole number is not divisible by 2.

   B. If a whole number is divisible by 2, then it is even.

   C. If a whole number is even, then it is not divisible by 2.

   D. For a whole number to be even it is sufficient that it is divisible by 2.

11. State whether the following statements are atomic or compound :
    (1) All natural numbers are integers.
    (2) If the mountain is high, then the sea is deep.
    (3) Integers are not rational numbers.
    (4) An integer is called a prime if it has no proper factor.
    (5) An integer having proper factors is said to be composite.
12. Write each sentence in the conditional form : (Using 'If .... then,')
    (a) All rational numbers are real numbers.
    (b) Freezing water expands.
    (c) A positive integer having no proper divisor is a prime.
    (d) Two sides of an isosceles triangle are equal.
13. When it does not rain but grass grows, what is the truth value of the statement "If it rains, then grass grows".

ANSWERS

1. (a) Statement, (b) Not a statement, (c) Not a statement, (d) Statement, (e) Statement, (f) Stateme
2. (i) $\sim p \wedge q$, (ii) $p \wedge \sim q$ (iii) $\sim p \wedge \sim q$ (iv) $p \vee (q \wedge \sim p)$.
3. (i) T, (ii) T, (iii) F, (iv) T, (v) T.
4. (i)

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

(ii)

p	q	r	~r	$p\Leftrightarrow q$	$\sim r\Rightarrow p$	$(p\Leftrightarrow q)\wedge(\sim r\Rightarrow p)$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	T	F	T	F	F	F
F	F	T	F	T	T	T
F	F	F	T	T	F	F

(iii)

p	q	r	~q	$p\vee\sim q$	$(p\vee\sim q)\wedge r$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	F	T	T	T
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	T

(iv)

p	q	r	$p\Rightarrow q$	$q\Rightarrow r$	$(p\Rightarrow q)\wedge(q\Rightarrow r)$	$p\Rightarrow r$	Given statement
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

5. (a) 5 is not a rational number, (b) 3 is a prime, (c) Some integers are not rational numbers, (d) All natural numbers are integers, (e) It is not true that a triangle has four sides, (f) Man is immortal, (g) Water is cold but fire is not hot, (h) Kumar or Kanta is not intelligent, (i) There is no student who never reads, (j) some students are dishonest, (k) It is not the case that every integer is either positive or negative, (l) There is a will and there is not a way.

6. (i) $(\sim p \wedge \sim q) \vee \sim r$, (ii) $(p \wedge q) \wedge \sim r$, (iii) $\sim p \vee (q \wedge \sim r)$

7. (a) Tautology, (b) Tautology, (c) Tautology, (d) Contradiction, (e) Tautology, (f) Neither tautology nor contradiction, (g Tautology, (h) Tautology, (i) Tautology.

8. C, 9. D 10. B 11. (1) & (5) are atomic ; (2), (3) and (4) are compound.

12. (a) If a number is rational then it is a real number.

(b) If water freezes, then it expan

(c) If a positive in ger has no proper divisor, then it is a prime.

(d) If a triangle is issosceles, then two of its sides are equ

13. T.

Chapter–4

LOGARITHMS

4.1. Introduction :

We are familiar with powers and roots of real numbers. In this chapter, we shall make an important discussion associated with the powers of positive real numbers. Let us first recall some facts associated with powers of positive numbers. The symbol a^m is usually read as 'a raised to the power m' or briefly as 'a to the power m'. Here a is called the base, m, the index or exponent and a^m, a power. You may recall from earlier classes the following laws called the 'laws of indices' or 'laws of exponents' :

(i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$ (iii) $a^m \times b^m = (ab)^m$

4.2. Definition of Logarithm :

For a positive real number a other than 1, if $a^m = x$, then we define that 'logarithm of x to the base a is m' and we use the notation $\log_a x = m$.

Here, 'a' is called the base of the logarithm, 'm' the value of the logarithm and 'log' stands for 'logarithm'. We read "$\log_a x$" as "logarithm of x to the base a" or briefly as "log of x to the base a".

Thus, the logarithm of a given number x to a given positive base $a(\neq 1)$ is the exponent of the power to which a is to be raised in order to equal x.

Observe that the two statements "$a^m = x$" and "$\log_a x = m$" are equivalent i.e. they have the same meaning. And from this fact we deduce the following relations :

(i) $a^{\log_a x} = x$ (ii) $\log_a a^m = m.$

Note : In the notation "$\log_a x$" the base a is a positive number different from 1 and x being equal to a power of the positive number a, is also a positive number.

From the definition, we can see that

$\log_2 16 = 4$ $\because\ 2^4 = 16$

and $\log_4 16 = 2$ $\because\ 4^2 = 16$

$\therefore$ $\log_2 16 \neq \log_4 16$

Thus, the logarithms of the same number with respect to different bases are different. Therefore, in each logarithm the base should be mentioned. However, when a relation involving logarithms is true for any permissible base or when the base is already known, the suffix denoting the base may be omitted.

For any positive number $a(\neq 1)$, we know that $a^0 = 1$ and $a^1 = a$. So, $\log_a 1 = 0$ and $\log_a a = 1$. Stating precisely, the logarithm of 1 to any positive base (other than 1) is zero and the logarithm of the base itself is 1.

Example 1. Find the logarithm of 6 to the base $\sqrt[3]{6}$.

Solution : Let $\log_{\sqrt[3]{6}} 6 = x$

Then $\left(\sqrt[3]{6}\right)^x = 6$

$\Rightarrow \left(6^{\frac{1}{3}}\right)^x = 6$

$\Rightarrow 6^{\frac{x}{3}} = 6$

$\Rightarrow \frac{x}{3} = 1$ $(\because$ bases are the same$)$

$\therefore x = 3.$

i.e. logarithm of 6 to the base $\sqrt[3]{6}$ is 3.

Example 2. Find the base if the logarithm of 3 is – 1.

Solution : Let a be the base.

Then $\log_a 3 = -1$

$\Rightarrow a^{-1} = 3$

$\Rightarrow \frac{1}{a} = 3$

$\Rightarrow a = \frac{1}{3}$

i.e. the base is $\frac{1}{3}$.

Example 3. If $\log_a x = m$, show that $\log_{\frac{1}{a}} x = -m$

Solution : $\log_a x = m$

$\Rightarrow a^m = x$

$$\Rightarrow \quad \left\{\left(\frac{1}{a}\right)^{-1}\right\}^{m} = x$$

$$\Rightarrow \quad \left(\frac{1}{a}\right)^{-m} = x$$

$$\Rightarrow \quad \log_{\frac{1}{a}} x = -m.$$

4.3. Laws of Logarithms :

There are basic laws of logarithms and they are deduced directly from the laws of exponents as follows :

Law 1 : $\log_a(m\times n) = \log_a m + \log_a n$

Proof : Let $\log_a m = x$ and $\log_a n = y$.

Then $a^x = m$ and $a^y = n$

$\therefore \quad a^x \times a^y = m\times n$

$\Rightarrow \quad a^{x+y} = m\times n$

$\Rightarrow \quad \log_a(m\times n) = x + y$ (by definition)

i.e. $\log_a(m\times n) = \log_a m + \log_a n$

Law 2 : $\log_a \frac{m}{n} = \log_a m - \log_a n$

Proof : Let $\log_a m = x$ and $\log_a n = y$.

Then $a^x = m$ and $a^y = n$

$$\therefore \quad \frac{a^x}{a^y} = \frac{m}{n}$$

$$\Rightarrow \quad a^{x-y} = \frac{m}{n}$$

$$\therefore \quad \log_a \frac{m}{n} = x - y \qquad \text{(by definition)}$$

i.e. $\log_a \frac{m}{n} = \log_a m - \log_a n$

Law 3 : $\log_a m^n = n\log_a m$

Proof : Let $\log_a m = x$

Then $a^x = m$

$\therefore \quad (a^x)^n = m^n$

$\Rightarrow \quad a^{nx} = m^n$

$\therefore \quad \log_a m^n = nx$

i.e. $\log_a m^n = n\log_a m$

The first three laws of logarithm (given above) are utilized, in the case of numbers, so as the process of

(i) multiplication is replaced by addition of their logarithms

(ii) division is replaced by subtraction of their logarithms

and (iii) involution or evolution i.e. raising to power or extraction of roots is replaced by multiplication.

Law 4 : (Change of base) :

$$\log_a m = \log_b m \times \log_a b = \frac{\log_b m}{\log_b a}$$

Proof : Let $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.

Then $a^x = m$, $b^y = m$ and $a^z = b$.

$\therefore \quad a^x = b^y$

$\Rightarrow \quad a^x = (a^z)^y$

$\Rightarrow \quad a^x = a^{yz}$

$\Rightarrow \quad x = yz$

i.e. $\log_a m = \log_b m \times \log_a b$ .... (i)

Since this relation holds for any positive number *m*, therefore we can replace *m* by *a* and obtain

$$\log_a a = \log_b a \times \log_a b$$

$\Rightarrow \quad 1 = \log_b a \times \log_a b$

$\Rightarrow \quad \log_a b = \dfrac{1}{\log_b a}$ .... (ii)

From (i) and (ii), we obtain

$$\log_a m = \log_b m \times \log_a b = \frac{\log_b m}{\log_b a}$$

This particular law, called the law of change of base in logarithms enables us to transform the system of logarithms with regard to a given base to a new system with a new base.

Given system of logarithms to a base, say b we can obtain a new system of logarithms to a desired base, say a simply by multiplying each logarithm by $\log_a b$. This constant factor $\log_a b$ is called the modulus of the system of logarithms to the base b with respect to the system of logarithms to the base a.

Example 1. Rewrite using logarithm the statement $\left(2\sqrt{3}\right)^4 = 144$

Solution : $\log_{2\sqrt{3}} 144 = 4$.

Example 2. Rewrite "$\log_3 81 = 4$" in the exponential form.

Solution : $3^4 = 81$.

Example 3. Show that $x^{\log_a y} = y^{\log_a x}$

Solution : Let $\log_a x = m$ and $\log_a y = n$

Then $a^m = x$ and $a^n = y$

Now,
$$\begin{aligned} x^{\log_a y} &= \left(a^m\right)^n \\ &= a^{mn} \\ &= \left(a^n\right)^m \\ &= y^m \\ &= y^{\log_a x}. \end{aligned}$$

Example 4. Express the logarithm of $\dfrac{\sqrt[3]{m}}{n^2\sqrt{p^5}}$ in terms of $\log m$, $\log n$ and $\log p$ (logarithms having the same base).

Solution : $\log\frac{\sqrt[3]{m}}{n^2\sqrt{p^5}} = \log\frac{m^{1/3}}{n^2 p^{5/2}}$

$= \log m^{1/3} - \log\left(n^2 p^{5/2}\right)$

$= \frac{1}{3}\log m - \left(\log n^2 + \log p^{5/2}\right)$

$= \frac{1}{3}\log m - \left(2\log n + \frac{5}{2}\log p\right)$

$= \frac{1}{3}\log m - 2\log n - \frac{5}{2}\log p.$

Example 5. Show that $3\log\frac{36}{25} + 3\log\frac{2}{9} - 2\log\frac{16}{125} = \log 2$

Solution : $3\log\frac{36}{25} + 3\log\frac{2}{9} - 2\log\frac{16}{125}$

$= 3(\log 36 - \log 25) + 3(\log 2 - \log 9) - 2(\log 16 - \log 125)$

$= 3\log 36 - 3\log 25 + 3\log 2 - 3\log 9 - 2\log 16 + 2\log 125$

$= 3\log(2^2 \times 3^2) - 3\log 5^2 + 3\log 2 - 3\log 3^2 - 2\log 2^4 + 2\log 5^3$

$= 3(\log 2^2 + \log 3^2) - 3\times 2\log 5 + 3\log 2 - 3\times 2\log 3 - 2\times 4\log 2 + 2\times 3\log 5$

$= 6\log 2 + 6\log 3 - 6\log 5 + 3\log 2 - 6\log 3 - 8\log 2 + 6\log 5$

$= (6+3-8)\log 2$

$= \log 2.$

Example 6. If $a > 1$ and $x > y > 0$, show that

(i) $\log_a x > \log_a y$

(ii) $\log_{1/a} x < \log_{1/a} y$

Solution : (i) Since $a > 1$ therefore $a^p > 1$ when $p > 0$ and $a^p < 1$ when $p < 0$.

Let $\log_a x = m$ and $\log_a y = n$

Then $a^m = x$ and $a^n = y$

$$\therefore \quad a^{m-n} = \frac{a^m}{a^n} = \frac{x}{y} > 1$$

This implies that $m - n > 0$

i.e. $m > n$

i.e. $\log_a x > \log_a y$

(ii) $\log_a x > \log_a y$ {by (i)}

$\Rightarrow -\log_a x < -\log_a y$

$\Rightarrow \log_{1/a} x < \log_{1/a} y$ (By example 3 of §4·2)

Example 7. Show that $\log_{10} 2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$.

Solution : We have

$8 < 10 < 16$

$\Rightarrow \quad 2^3 < 10 < 2^4$

Taking logarithms to the base 10, we get (since $10 > 1$)

$$\log_{10} 2^3 < \log_{10} 10 < \log_{10} 2^4$$

$$\Rightarrow \quad 3\log_{10} 2 < 1 < 4\log_{10} 2$$

Thus, $3\log_{10} 2 < 1$ so that $\log_{10} 2 < \frac{1}{3}$ .... (i)

and $1 < 4\log_{10} 2$ so that $\frac{1}{4} < \log_{10} 2$ .... (ii)

Combining (i) and (ii), we obtain

$$\frac{1}{4} < \log_{10} 2 < \frac{1}{3}.$$

Example 8. If $\log_a x = 10$ and $\log_{6a}(32x) = 5$, find *a*.

Solution : We have $\log_a x = 10 \Rightarrow a^{10} = x$ .... (i)

and $\log_{6a}(32x) = 5 \Rightarrow (6a)^5 = 32x$ .... (ii)

Dividing (i) by (ii), we get

$$\frac{a^{10}}{(6a)^5}=\frac{x}{32x}$$

$$\Rightarrow \quad \frac{a^{10}}{6^5\times a^5}=\frac{1}{32}$$

$$\Rightarrow \quad \left(\frac{a}{6}\right)^5=\left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad \frac{a}{6}=\frac{1}{2}$$

$$\therefore \quad a=\frac{6}{2}=3.$$

Example 9. If $a^2+b^2=14ab$, show that $\log\left(\frac{a+b}{4}\right)=\frac{1}{2}(\log a+\log b)$

Solution : We have

$$a^2+b^2=14ab$$

$$\Rightarrow \quad a^2+b^2+2ab=14ab+2ab$$

$$\Rightarrow \quad (a+b)^2=16ab$$

$$\Rightarrow \quad \left(\frac{a+b}{4}\right)^2=ab$$

$$\therefore \quad \log\left(\frac{a+b}{4}\right)^2=\log(ab)$$

$$\Rightarrow \quad 2\log\left(\frac{a+b}{4}\right)=\log a+\log b$$

$$\Rightarrow \quad \log\frac{a+b}{4}=\frac{1}{2}(\log a+\log b)$$

Example 10. If $m=\log_a(bc)$, $n=\log_b(ca)$, $p=\log_c(ab)$, prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1$$

Solution : We have

$$m=\log_a(bc)$$

$$\Rightarrow \quad m+1=\log_a(bc)+1=\log_a(bc)+\log_a a \qquad (\because \log_a a=1)$$
$$=\log_a(abc)$$

Similarly, $n=\log_b(ca)\Rightarrow n+1=\log_b(abc)$

and $p=\log_c(ab)\Rightarrow p+1=\log_c(abc)$

$$\therefore \frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=\frac{1}{\log_a(abc)}+\frac{1}{\log_b(abc)}+\frac{1}{\log_c(abc)}$$
$$=\log_{abc} a+\log_{abc} b+\log_{abc} c$$
$$=\log_{abc}(abc)$$
$$=1.$$

Example 11. If $\log_{10} 2 = p$, express $\log_5 80$ in terms of p.

Solution : We have

$$\log_{10} 2 = p$$

$$\therefore \quad \log_{10} 5=\log_{10}\left(\frac{10}{2}\right)$$
$$=\log_{10} 10-\log_{10} 2$$
$$=1-\mathrm{p}$$

Now, $\log_5 80=\log_5(2^4\times 5)=\log_5 2^4+\log_5 5$

$$=4\log_5 2+1$$
$$=4\times\frac{\log_{10} 2}{\log_{10} 5}+1$$
$$=\frac{4p}{1-p}+1=\frac{3p+1}{1-p}.$$

EXERCISE 4.1

1. Rewrite the following using logarithms :

(i) $10^3 = 1000$ (ii) $4^{-2} = \frac{1}{16}$

(iii) $10^{-3} = 0{\cdot}001$ (iv) $(1000)^{1/3} = 10$.

2. Rewrite in the exponential form :

(i) $\log_{10}({\cdot}01) = -2$ (ii) $\log_2 64 = 6$

(iii) $\log_3 243 = 5$ (iv) $\log_m l = p$.

3. Find the logarithm of

(i) 324 to the base $3\sqrt{2}$ (ii) 400 to the base $2\sqrt{5}$

(iii) $\sqrt{3}$ to the base $\sqrt[5]{3}$ (iv) $\cdot$008 to the base $\sqrt{5}$.

4. Find the base if the logarithm of

(i) 625 is 4 (ii) 32 is $\frac{4}{3}$ (iii) 20 is 2.

5. Show that log (1 × 2 ×3) = log 1 + log 2 + log 3. Is it true for any three positive numbers *m, n, p* (instead of 1, 2, 3) ?

6. Prove that

(i) $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80} = 1$

(ii) $\log\frac{14}{15} + \log\frac{28}{27} + \log\frac{405}{196} = \log 2$

(iii) $4\log 2 + 3\log 3 - 2\log 12 = \log 3$

(iv) $\log_2\frac{448}{625} = 6 + \log_2 7 - 4\log_2 5$

(v) $\log_2 \log_2 \log_2 16 = 1$

(vi) $\log_3 \log_2 \log_2 256 = 1$

(vii) $7\log\frac{15}{16} + 6\log\frac{8}{3} + 5\log\frac{2}{5} + \log\frac{32}{25} = \log 3$

7. If $\log(m+n) = \log m + \log n$, express m in terms of n.

8. If x be the logarithm of a number to the base $2\sqrt{2}$, show that the logarithm of the number to the base 2 is $3x$.

9. Prove that $\log_{a^p}(x^p) = \log_a x$ for any non-zero real number p.

10. Find a, if $\frac{\log(5a-6)}{\log a} = 2$

11. Prove that

(i) $\log_b a \times \log_c b \times \log_a c = 1$

(ii) $\log_a b \times \log_b c \times \log_c a = 1$

(iii) $\log_b a \times \log_c b \times \log_d c = \log_d a$

12. (i) If $a^2 + b^2 = 7ab$, show that $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

(ii) If $a^2 + b^2 = 27ab$ and $a > b$, show that $\log\frac{a-b}{5} = \frac{1}{2}(\log a + \log b)$

13. If $a^{2-x}b^{3x} = a^{x+3}b^x$, show that $x = \frac{\log a}{2(\log b - \log a)}$

14. If $\log(x^2y^3) = a$ and $\log\frac{x}{y} = b$, find $\log x$ and $\log y$ in terms of a and b.

15. Prove that $\log_a x \times \log_b y = \log_b x \times \log_a y$

16. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, show that

(i) $abc = 1$ (ii) $a^a b^b c^c = 1$

17. If $\log_3 15 = p$, show that $\log_5 675 = \dfrac{2p+1}{p-1}$

18. Prove that $\log_5 3 < \dfrac{8}{5}$

19. Prove that $\log_{75} 135 = \dfrac{2p-1}{3-p}$ where $p = \log_{15} 45$

20. Solve

(i) $\log_{10} x + \log_{10}(x-15) = 2$

(ii) $\log_2 x + \log_2 \dfrac{x}{16} = \log_2 \dfrac{x}{64}$.

ANSWERS

1. (i) $\log_2 1000 = 3$ (ii) $\log_4 \dfrac{1}{16} = -2$ (iii) $\log_{10} 0{\cdot}001 = -3$ (iv) $\log_{1000} 10 = \dfrac{1}{3}$

2. (i) $10^{-2} = {\cdot}01$ (ii) $2^6 = 64$ (iii) $3^5 = 243$ (iv) $m^p = l$.

3. (i) 4 (ii) 4 (iii) $\dfrac{5}{4}$ (iv) -6.

4. (i) 5 (ii) $\sqrt[4]{2^{15}}$ (iii) $2\sqrt{5}$.

5. No. 7. $\dfrac{n}{n-1}$ 10. $a = 2, 3$

14. $\log x = \dfrac{1}{5}(a+3b)$, $\log y = \dfrac{1}{5}(2b-a)$

20. (i) 20 (ii) 4, 8.

Chapter–5

TRIGONOMETRIC RATIOS

5.1. Introduction :

The main purpose in trigonomety, is to solve the following problem : If some sides and angles of a triangle are known, how do we find the remaining sides and angles ?

This problem is solved by using some ratios of the sides of a right triangle with respect to its acute angles called *trigonometric ratios of angles*. There are six such trigonometric ratios. They are *sine, cosine, tangent, cosecant, secant* and *cotangent* of an angle θ and they are shortly written as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ respectively.

5.2. Sine, Cosine, Tangent of an angle :

In Fig 5.1, θ is a given acute angle ($\angle$YAX). We take a point P on $\overrightarrow{AY}$ and drop the perpendicular PB on $\overrightarrow{AX}$. Then we have right ΔPAB in which $m\angle PAB=\theta$. Then the ratio $\dfrac{PB}{AP}$ is called the *sine of angle* θ.

Thus, $\sin\theta=\dfrac{PB}{AP}$

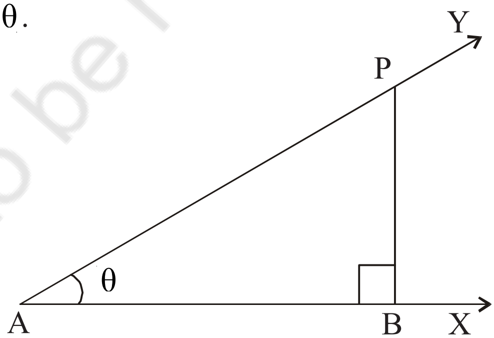


Fig. 5.1

Since in the rt. angled ΔPAB, the side $\overline{PB}$ is opposite to angle θ and $\overline{AP}$ is the hypotenuse, we actually have,

$$\boxed{\sin\theta=\frac{\text{opposite side}}{\text{hypotenuse}}}$$

However, we might well ask, what happens if we choose P somewhere else on $\overrightarrow{AY}$? If we take a different position for P, the lengths of $\overline{PB}$ and $\overline{AP}$ will change but the ratio $\dfrac{PB}{AP}$ will remain the same as before. We take this result for granted.

Going back to the right ΔPAB, in which $m\angle PAB=\theta$, we define two more trigonometric ratios of θ as follows :–

$$\text{cosine of }\theta=\frac{AB}{AP},\quad \text{tangent of }\theta=\frac{PB}{AB}$$

or, in short, $\cos\theta=\dfrac{AB}{AP}$, $\tan\theta=\dfrac{PB}{AB}$

Since in ΔPAB, $\overline{PB}$ is the adjacent side, in relation to angle θ, we have, $\boxed{\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}}$, $\boxed{\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}}$

It may be noted that like $\sin\theta$, $\cos\theta$ and $\tan\theta$ also depend only on the angle θ and not on the size of the rt. triangle.

Note : "$\sin\theta$" is an abbreviation for "sine of angle θ"; it is not the product of sin and θ.

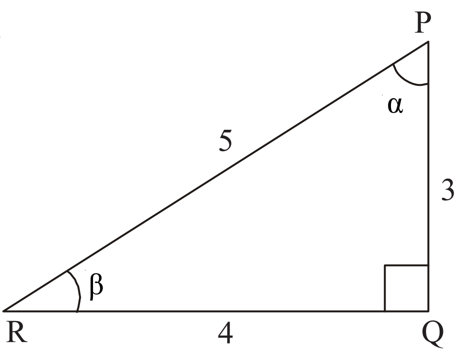


Fig. 5.2

Example 1 : In ΔPQR, Q is a right angle, PQ = 3 cm. and QR = 4 cm. If $m\angle P = \alpha$, $m\angle R = \beta$, find $\sin\alpha$, $\cos\alpha$ $\tan\alpha$, $\sin\beta$, $\cos\beta$ and $\tan\beta$.

Solution : In Fig 5.2, from rt. ΔPQR by Pythagoras theorem,

$$RP^2 = PQ^2 + RQ^2$$

$$\Rightarrow \quad RP = \sqrt{9+16} = 5 \text{ cm.}$$

For α, the opposite side is of length 4 cm and the adjacent side of 3 cm. For β, the opposite side is of length 3 cm and adjacent side of 4 cm.

So, $\sin\alpha = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{4}{5}$, $\cos\alpha = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{3}{5}$,

$$\tan\alpha = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{4}{3}$$

Similarly, $\sin\beta = \dfrac{3}{5}$, $\cos\beta = \dfrac{4}{5}$ and $\tan\beta = \dfrac{3}{4}$.

EXERCISE 5.1

In the ΔABC, $\angle$A is right angle. The lengths of the sides are given in cm in each of the following : find sin B, cos B, tan B, sin C, cos C and tan C.

1. $BC = \sqrt{2}$, $AB = AC = 1$
2. $AB = 5$, $AC = 12$, $BC = 13$
3. $AB = 3$, $AC = 4$, $BC = 5$
4. $AB = 20$, $AC = 21$, $BC = 29$

5.3. Relation between $\sin\theta$, $\cos\theta$ and $\tan\theta$:

The ratios $\sin\theta$, $\cos\theta$, $\tan\theta$ of an angle θ are very closely related. If any one of them is known the other two can easily be calculated. In Fig. 5.3,

$$\sin\theta = \frac{PR}{QR},\ \cos\theta = \frac{QR}{QP}$$

and $\tan\theta = \dfrac{PR}{QP} = \dfrac{PR/QP}{QR/QP} = \dfrac{\sin\theta}{\cos\theta}$

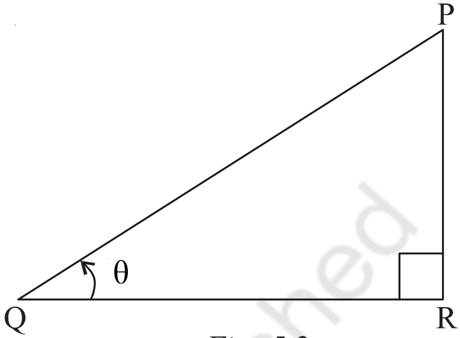


Fig. 5.3

Example 2 : If $\sin\theta = \frac{3}{5}$, find $\cos\theta$ and $\tan\theta$.

Solution : In Fig 5.4, ABC is a right triangle, in which $\angle ABC = \theta$.

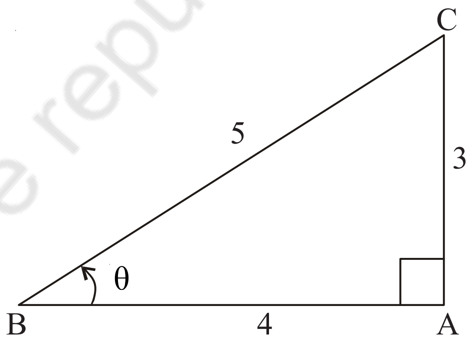


Fig. 5.4

Given that $\sin\theta = \frac{3}{5}$,

$\Rightarrow$ $AC = 3$ and $BC = 5$.

By Pythagoras theorem,

$$AB = \sqrt{BC^2 - AC^2} = \sqrt{25-9} = 4$$

$$\therefore\ \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{4}{5}$$

also, $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{3}{4}$

Example 3 : If $\cos A = \frac{1}{\sqrt{3}}$, find sin A and tan A.

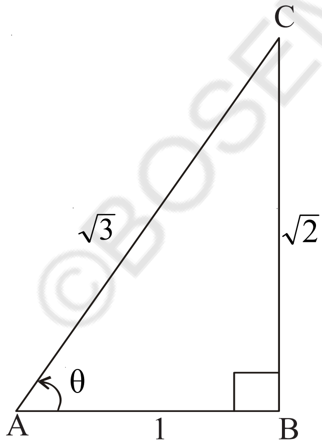


Fig. 5.5

Solution : In Fig. 5.5, ABC is a right Δ.

Given that $\cos A = \frac{1}{\sqrt{3}}$

$\Rightarrow$ $AC = \sqrt{3}$ and $AB = 1$

By Pythagoras theorem,

$$BC = \sqrt{AC^2 - AB^2} = \sqrt{3-1} = \sqrt{2}$$

$$\therefore\ \sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{\frac{2}{3}}$$

and $\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$

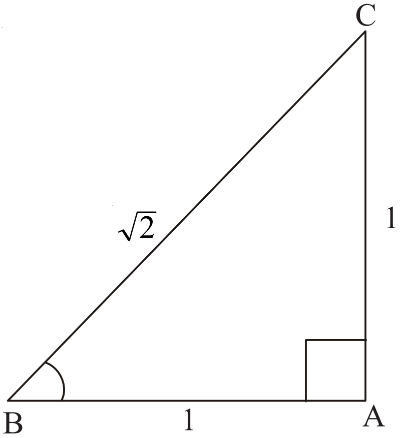


Fig. 5.6

Example 4 : If tan B=1, find sin B and cos B

Solution : In Fig 5.6, ABC is a right Δ. Given that tan B=1.

$$\Rightarrow AB = AC = 1 \ \ \text{(say)}$$

By Pythagoras theorem, $BC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\therefore \ \sin B = \frac{AC}{BC} = \frac{1}{\sqrt{2}} \text{ and } \cos B = \frac{AB}{BC} = \frac{1}{\sqrt{2}}$$

EXERCISE 5.2

In each of the following, one of the three trigonometric ratios (sine, cosine and tangent of an angle) is given. Find the other two ratios.

1. $\sin A = \frac{2}{3}$ 2. $\cos B = \frac{2}{5}$ 3. $\tan C = \frac{12}{5}$

4. $\sin \alpha = \frac{4}{5}$ 5. $\cos \beta = \frac{3}{5}$ 6. $\tan \theta = 10$.

5.4. The other Trigonometric Ratios :

Of the six trigonometric ratios, we have already defined sine, cosine and tangent of an angle θ. We shall now define the remaining three ratios, namely, cosecant, secant and cotangent of an angle θ. For any acute angle θ,

$$\boxed{\operatorname{cosec}\theta = \frac{1}{\sin\theta}}, \quad \boxed{\sec\theta = \frac{1}{\cos\theta}}, \quad \boxed{\cot\theta = \frac{1}{\tan\theta}}$$

It is obvious that out of the six trigonometric ratios of an angle, if any one is known, all the others can be calculated.

Example 5 : If $\sec\theta = 2$, find the other five trigonometric ratios of θ.

Solution : Since $\sec\theta = 2$, $\therefore \cos\theta = \frac{1}{2}$

In Fig 5.7, ABC is a rt.Δ in which $m\angle ABC = \theta$.

Then $AB = 1, BC = 2 \ \therefore \ AC = \sqrt{4-1} = \sqrt{3}$

Now, $\sin\theta = \frac{AC}{BC} = \frac{\sqrt{3}}{2}, \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

$$\tan\theta = \frac{AC}{AB} = \sqrt{3}, \ \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$$

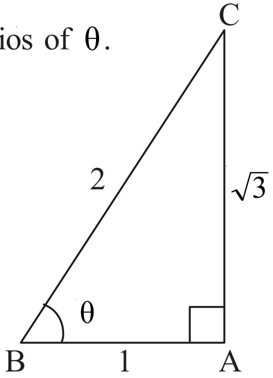


Fig. 5.7

Example 6 : In a ΔABC rt. angled at C, $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \sqrt{3}$, show that cos A cos B – sin A sin B=0.

Solution : In Fig 5.8, by the given condition $AC = \sqrt{3}$ and $BC = 1$.

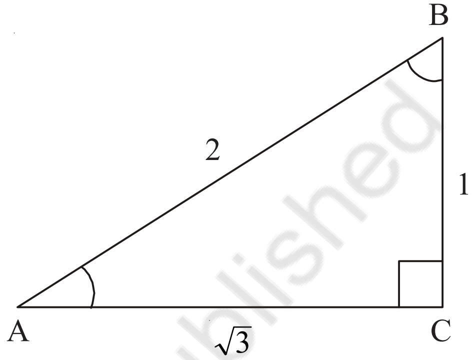


Fig. 5.8

$$\therefore \quad AB = \sqrt{AC^2 + BC^2} = \sqrt{3+1} = 2$$

So, we have, $\sin A = \frac{1}{2}$, $\sin B = \frac{\sqrt{3}}{2}$

$$\cos A = \frac{\sqrt{3}}{2}, \ \cos B = \frac{1}{2}$$

$$\therefore \quad \cos A \cos B - \sin A \sin B = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

Hence, cos A cos B – sin A sin B=0.

EXERCISE 5.3

1. If $\operatorname{cosec}\theta = 2$, find the other five trigonometric ratios of θ.

2. If $\sec\theta = \frac{2}{\sqrt{3}}$, find the other five trigonometric ratios of θ.

3. In a ΔABC rt. angled at C, if $\tan A = \sqrt{3}$ and $\tan B = \frac{1}{\sqrt{3}}$,

   show that $\sin A \cos B + \cos A \sin B = 1$.

4. If tan A=1 and tan B=$\sqrt{3}$, evaluate cos A cos B – sin A sin B.
   [Here, A and B are not the angles of the same triangle.]

5. If $\sec\theta = \frac{5}{4}$, verify that $\tan\theta = \sin\theta \ \sec\theta$.

6. If $\cos B = \frac{\sqrt{3}}{2}$, show that $3 \sin B - 4 \sin^3 B = 1$.
   [Hint : $(\sin B)^3$ is written as $\sin^3 B$.]

5.5. Some Useful Identities :

In Fig. 5.9, PAM is a rt. Δ where $m\angle \text{PAM} = \theta$. We represent hypotenuse $\overline{\text{AP}}$, adjacent side $\overline{\text{AM}}$ and opposite side $\overline{\text{PM}}$ respectively by *r*, *x* and *y*.

Then $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$

$\therefore\ \sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$

Hence, $\boxed{\sin^2\theta + \cos^2\theta = 1}$ ....(i)

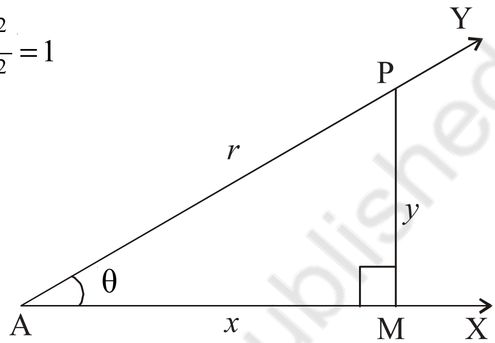


Fig. 5.9

Dividing (i) throughout by $\cos^2\theta$,we get,.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

i.e. $\boxed{\tan^2\theta + 1 = \sec^2\theta}$

Again dividing (i) the throughout by $\sin^2\theta$ we get,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

i.e. $\boxed{1 + \cot^2\theta = \text{cosec}^2\theta}$

Example 7 : Prove that :

(i) $\tan A + \cot A = \text{cosec}\, A \sec A$ (ii) $\frac{1}{1+\tan^2\theta} + \frac{1}{1+\cot^2\theta} = 1$

(iii) $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution : (i) L.S. $= \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A} = \frac{1}{\cos A \sin A}$$

$$= \frac{1}{\sin A}\cdot\frac{1}{\cos A} = \text{cosec}\, A\,.\sec A = \text{R.S.}$$

(ii) L.S. $= \frac{1}{1+\tan^2\theta} + \frac{1}{1+\cot^2\theta} = \frac{1}{\sec^2\theta} + \frac{1}{\text{cosec}^2\theta}$

$= \cos^2\theta + \sin^2\theta = 1 =$ R.S.

(iii) R.S. $= 1 + 2\sin\theta\cos\theta$

$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$= (\sin\theta + \cos\theta)^2 =$ L.S.

EXERCISE 5.4

Prove that (Q. No. 1 –13)

1. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$ 2. $1 - \sin^2\theta = \frac{1}{1 - \tan^2\theta}$ 3. $2\cos^2\theta - 1 = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

4. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

5. $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$

6. $\sqrt{\sec^2\theta - 1} = \sin\theta\sec\theta$

7. $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta$

8. $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta.\text{cosec}^2\theta$

9. $\tan^2\theta - \sin^2\theta = \tan^2\theta.\sin^2\theta$

10. $\frac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$

11. $\frac{\sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta}{\sin\theta}$

12. $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$

13. $\frac{1 + \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 - \sin\theta}$

14. Eliminate θ from the equations :–

(i) $x = r\cos\theta,\ y = r\sin\theta$ (ii) $x = a\sec\theta, y = b\tan\theta$ (iii) $x = a\ \text{cosec}\ \theta, y = b\cot\theta$

(iv) $\sin\theta + \cos\theta = a$
$\tan\theta + \cot\theta = b$

(v) $x\cos\theta + y\sin\theta = 3$
$y\cos\theta - x\sin\theta = 4$

15. If $\tan\theta + \sec\theta = x$, show that $\sin\theta = \frac{x^2 - 1}{x^2 + 1}$.

16. If $\tan\theta = \frac{a}{b}$, show that $\frac{a\sin\theta - b\cos\theta}{a\sin\theta + b\cos\theta} = \frac{a^2 - b^2}{a^2 + b^2}$.

17. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

5.6. Trigonometric Ratios of Some standard Angles :

We have defined sin θ, cos θ, tan θ etc. for any acute angle θ. But we have not yet found their values for even one specific θ. We can use our knowledge of geometry to find the values of the trigonometric ratios of some standard angles, namely, 30°, 60° and 45°.

5.7. Trigonometric Ratios of 30° :

We may recall that the measure of each angle of an equilateral triangle is 60°. Thus, the bisector of angle of such triangle makes with either side an angle of 30°.

In Fig. 5.10, ABC is an equilateral triangle with each side of length a. Let $\overline{AD}$ be perpendicular to $\overline{BC}$. Since the triangle is equilateral, $\overline{AD}$ is also the bisector of $\angle A$ and D is the mid point of $\overline{BC}$.

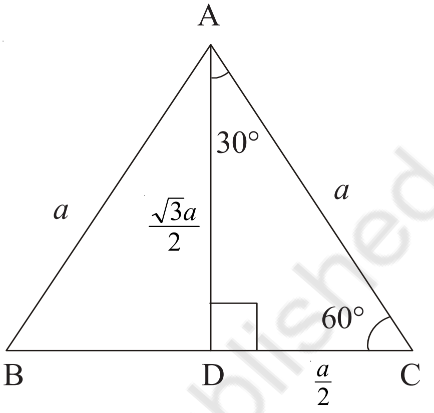


Fig. 5.10

$\therefore$ $DC = \frac{a}{2}$ and $\angle CAD = 30°$

In ΔADC $m\angle D = 90°$, hypotenuse $AC = a$.

So, by Pythagoras theorem,

$$AD^2 = AC^2 - DC^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\therefore \quad AD = \frac{\sqrt{3}}{2}a$$

Considering the rt. triangle ADC, where $m\angle CAD = 30°$,

we have, $\sin 30° = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{DC}{AC} = \frac{a/2}{a} = \frac{1}{2}$

$$\cos 30° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AD}{AC} = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{DC}{AD} = \frac{a/2}{\sqrt{3}a/2} = \frac{1}{\sqrt{3}}$$

$$\text{cosec}\, 30° = \frac{1}{\sin 30°} = 2$$

$$\sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$$

$$\cot 30° = \frac{1}{\tan 30°} = \sqrt{3}$$

5.8. Trigonometric Ratios of 60° :

Referring again to Fig. 5.10, in the rt. triangle ADC, where $m\angle ACD = 60°$, we have

$$\sin 60° = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AD}{AC} = \frac{\sqrt{3}a/2}{a} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{DC}{AC} = \frac{a/2}{a} = \frac{1}{2}$$

$$\tan 60° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AD}{DC} = \frac{\sqrt{3}a/2}{a/2} = \sqrt{3}$$

$$\text{cosec}\,60° = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}$$

$$\sec 60° = \frac{1}{\cos 60°} = 2$$

$$\cot 60° = \frac{1}{\tan 60°} = \frac{1}{\sqrt{3}}$$

5.9. Trigonometirc Ratios of 45° :

In a right triangle ABC with $m\angle C = 90°$; $m\angle A = m\angle B = 45°$. Consequently BC = AC.

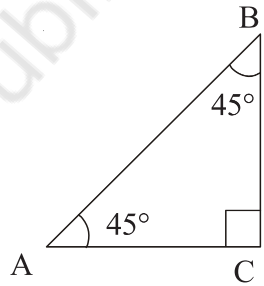


Fig. 5.11

Suppose BC = AC = a, then by Pythagoras theorem

$$AB^2 = BC^2 + AC^2 = a^2 + a^2 = 2a^2$$

$$\therefore\ AB = \sqrt{2}a$$

Considering the rt. triangle ABC, $m\angle A = 45°$,

we get, $$\sin 45° = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AC} = \frac{a}{a} = 1$$

Therefore, $$\text{cosec}\,45° = \frac{1}{\sin 45°} = \sqrt{2}$$

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$$

$$\cot 45° = \frac{1}{\tan 45°} = 1$$

5.10. Concept of Infinity ($\propto$) :

The symbol $\propto$ (called infinity), which we use to denote the result of dividing a positive number by zero (say $\frac{1}{0}$) is not a real number. This is an undefined number $\propto$, taken to be greater than every real number. Thus, $-\propto$ and $+\propto$ are beyond the extreme ends of the number line. The addition of a real number to $\propto$ would still be $\propto$. That is $x \in R$, $x + \propto = \propto$ and $x - \propto = x + (-\propto) = -\propto$.

For example, $5+\infty=\infty$, $10-\infty=-\infty$, $-4+\infty=+\infty$, $-4-\infty=-\infty$

Further, the product of ∞ by a number could still be ∞ and the division of a finite number by ∞ would give zero.

For Example, $\infty\times 5=\infty$ and $\frac{5}{\infty}=0$ etc.

5.11. Trigonometric Ratios of 0° and 90° :

We have defined $\sin\theta$, $\cos\theta$, $\tan\theta$ etc. for an acute angle θ, i.e. for θ such that $0°<\theta<90°$.

If $\theta=0°$ or if $\theta=90°$, we get separate definitions as follows :

(a) $\sin 0°=0, \cos 0°=1, \tan 0°=0,\ \sec 0°=1$ and $\operatorname{cosec} 0°=\infty,\ \cot 0°=\infty$.

(b) $\sin 90°=1, \cos 90°=0, \operatorname{cosec} 90°=1,\ \cot 90°=0, \sec 90°=\infty,\ \tan 90°=\infty$

TABLE 5.1

θ	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

Note : You may refer whenever necessary, the following table, 5.2 for the values of sine, cosine and tangent of some standard angles (of which the proof will be discussed in higher classes).

TABLE 5.2

θ	120°	135°	150°	180°
$\sin\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos\theta$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\theta$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Example 8 : Evaluate (i) cos 30° cos 60° – sin 30° sin 60°. (ii) $\sin^2 45° + \cos^2 45°$.

Solution : (i) Given expression $= \cos 30° \cos 60° - \sin 30° \sin 60°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

(ii) Given expression $= \sin^2 45° + \cos^2 45°$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

Example 9 : Verify (i) sin 60° = 2 sin 30° cos 30° (ii) $\tan 60° = \frac{2 \tan 30°}{1 - \tan^2 30°}$

Solution : (i) L.S. $= \sin 60° = \frac{\sqrt{3}}{2}$

R.S. $= 2 \sin 30° \cos 30° = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

∴ L.S. = R.S.

(ii) L.S. $= \tan 60° = \sqrt{3}$

$$\text{R.S.} = \frac{2 \tan 30°}{1 - \tan^2 30°} = \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}}$$

$$= \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}$$

∴ L.S. = R.S.

Example 10 : By putting A = 60° and B = 30°, examine whether the following relation holds good.

sin (A + B) = sin A cos B + cos A sin B.

Solution : L.S. = sin (A + B) = sin (60° + 30°) = sin 90° = 1

R.S. = sin A cos B + cos A sin B

= sin 60° cos 30° + cos 60° sin 30°

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{3+1}{4} = 1$$

L.S = R.S. i.e. the relation holds good.

EXERCISE 5.5

1. Evaluate the following expressions.

   (a) $^2\ 30° - \tan^2 30°$ (b) $\text{cosec}^2\ 45° - \cot^2 45°$ (c) $\sin 60°\ \cos 30° + \cos 60°\ \sin 30°$

   (d) $90°\ \cos 45° - \sin 90°\ \sin 45°$ (e) $\dfrac{2\tan 30°}{1-\tan^2 30°}$

2. Verify each of the following :

   (a) $90° = 1 - 2\sin^2 45° = 2\cos^2 45° - 1$ (b) $\dfrac{\tan 60° - \tan 30°}{1+\tan 60° \tan 30°} = \tan 30°$

   (c) $\dfrac{\cot 30° \cot 60° - 1}{\cot 60° + \cot 30°} = 0$ (d) $\dfrac{1-\sin 60°}{\cos 60°} = \dfrac{1-\tan 30°}{1+\tan 30°}$

   (e) $\dfrac{\cos 30° + \sin 60°}{1+\sin 30° + \cos 60°} = \cos 30°$

3. By putting A = 60° and B = 30°, examine which of the following relations hold good.

   (a) $\sin \quad - B) = \sin A \cos B - \cos A \sin B$ (b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

   (c) $\tan\ (A \quad = \tan A + \tan B$ (d $\quad A+B) = \sin A + \sin B$.

ANSWERS

EXERCISE 5.1

1. $\sin B = \sin C = \cos B = \cos C = \dfrac{1}{\sqrt{2}}$, $\tan B = \tan C = 1$.

2. $\sin B = \dfrac{12}{13}$, $\cos B = \dfrac{5}{13}$, $\tan B = \dfrac{12}{5}$, $\sin C = \dfrac{5}{13}$, $\cos C = \dfrac{12}{13}$, $\tan C = \dfrac{5}{12}$

3. $\sin B = \dfrac{4}{5}$, $\cos B = \dfrac{3}{5}$, $\tan B = \dfrac{4}{3}$, $\tan C = \dfrac{3}{4}$

4. $\sin B = \dfrac{21}{29}$, $\cos B = \dfrac{20}{27}$, $\tan B = \dfrac{21}{20}$, $\sin C = \dfrac{20}{29}$, $\cos C = \dfrac{21}{29}$, $\tan B = \dfrac{20}{21}$

EXERCISE 5.2

1. $\cos A = \dfrac{\sqrt{5}}{3}$, $\tan A = \dfrac{2}{\sqrt{5}}$ 2. $\sin B = \dfrac{\sqrt{21}}{5}$, $\tan B = \dfrac{\sqrt{21}}{2}$

3. $\sin C = \frac{12}{13},\ \cos C = \frac{5}{13}$ 4. $\cos\alpha = \frac{3}{4},\ \tan\alpha = \frac{4}{3}$ 5. $\sin\beta = \frac{4}{5},\ \tan\beta = \frac{4}{3}$

6. $\sin\theta = \frac{10}{\sqrt{101}},\ \cos\theta = \frac{1}{\sqrt{101}}$

EXERCISE 5.3

1. $\sin\theta = \frac{1}{2},\ \cos\theta = \frac{\sqrt{3}}{2},\ \sec\theta = \frac{2}{\sqrt{3}},\ \tan\theta = \frac{1}{\sqrt{3}},\ \cot\theta = \sqrt{3}$.

2. $\cos\theta = \frac{\sqrt{3}}{2},\ \sin\theta = \frac{1}{2},\ \operatorname{cosec}\theta = 2,\ \tan\theta = \frac{1}{\sqrt{3}},\ \cot\theta = \sqrt{3}$ 3. $\frac{1-\sqrt{3}}{2\sqrt{2}}$

EXERCISE 5.4

14. (i) $x^2 + y^2 = r^2$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(iv) $b(a^2 - 1) = 2$ (v) $x^2 + y^2 = 25$

EXERCISE 5.5

1. (a) 1 (b) 1 (c) 1 (d) $\frac{-1}{\sqrt{2}}$ (e) $\sqrt{3}$

3. (a) yes (b) yes (c) No (d) No

Chapter–6

PERMUTATIONS AND COMBINATIONS

6.1. Introduction :

The diagram in figure 6·1 represents Tomba's house, school, post office, market and the roads linking them. There are three roads from Tomba's house to the post office viz. A_1, A_2, A_3. From the post office to the school, there are two ways viz. B_1, B_2. Finally, there are four roads from the school to the market viz., C_1, C_2, C_3, C_4.

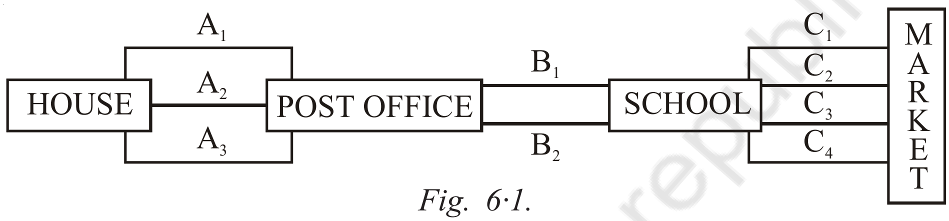


Fig. 6·1.

How many routes can Tomba follow from his house to the school ? First, he can follow road A_1 to reach post office and from the post office, he can follow either road B_1 or road B_2. It gives two possible routes : (A_1, B_1), (A_1, B_2). Similarly, the other possible routes are (A_2, B_1), (A_2, B_2), (A_3, B_1), (A_3, B_2). It gives us 6 i.e. 3×2 possible routes.

The total number of ways is 3×2 not $3 + 2$ because, for each of 3 roads from the house to the post office, there are 2 possible roads from the post office to the school. Thus, when there are 3 ways of performing an operation and 2 ways of performing another operation, then the total number of ways of performing these two operations is 3×2 i.e. 6.

Next, let us examine how many routes Tomba can follow from his house to the market. We have seen that there are 6 i.e. 3×2 ways in which Tomba can go from his house to the school and there are 4 roads from the school to the market. By a similar argument as in above, there are altogether $3 \times 2 \times 4$ routes from the house to the market such as (A_1, B_1, C_1), (A_1, B_1, C_2), (A_1, B_1, C_3) and so on [Write all the possible routes from the house to the market]. Thus, when there are three ways of performing an operation, two ways of performing a second operation and four ways of performing a third operation, then there are altogether $3 \times 2 \times 4$ ways of performing all the three operations. These results lead to the following principle.

Fundamental principle of counting or Multiplication principle :

'If an operation can be performed in m different ways, following which another operation can be performed in n different ways, then the total number of ways of performing both the operations is $m \times n$'.

The above principle can be generalised for any finite number of operations.

In the case of finding the total number of routes from Tomba's house to the school, the following operations are in succession :

(i) the operation of choosing a road from Tomba's house to post office.

(ii) the operation of choosing a road from post office to school.

Again, in the case of finding the number of routes from Tomba's house to the market, the following operations are in succession :

(i) the operation of choosing a road from Tomba's house to the post office.

(ii) the operation of choosing a road from the post office to the school.

(iii) the operation of choosing a road from the school to the market.

Example 1. Chaoba has three trousers of different colours and four shirts also of different colours. In how many ways, can he wear a trouser and a shirt ?

Solution : Since there are 3 trousers available, there are 3 ways in which a trouser can be chosen. Similarly, a shirt can be chosen in 4 ways. For every choice of a trouser, there are 4 choices of a shirt. Therefore, there are $4 \times 3 = 12$ possible pairs of a trouser and a shirt. Thus, he can wear a trouser and a shirt in 12 ways.

Example 2. Given 5 flags of different colours, how many different signals can be generated if each signal requires 2 flags, one below the other ?

Solution : Imagine that there are two vacant spaces to make a 2 flags signal. There will be as many 2 flags signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By the Multiplication principle, the total number of signals is $5 \times 4 = 20$.

6.2. Permutation :

Let there be n different things. Then an ordered arrangement of some or all of them is called a **permutation** of the things. e.g., for the three digits 1, 2, 3 each of the arrangements 123, 132, 213, 231, 312, 321 is called a permutation of the three digits taken all at a time. They are said to be different permutations because the digits are in different orders. 12, 21, 13, 31, 23, 32 are called permutations of the three digits taken two at a time while 1, 2, 3 are called permutations of the digits taken one at a time. From this discussion, we see that the total number of permutations of three things taken all at a time is 6, three things taken two at a time is 6 and that of three things taken one at a time is 3.

Definition : A permutation is an arrangement in a definite order of a number of things taken some or all at a time.

In other words, if from a set of n elements, $r(\leq n)$ elements are taken and arranged in a definite order, each arrangement is called a permutation of the n elements taken r at a time. The total number of permutations of n different objects taken $r(0<r\leq n)$ at a time and no object is repeated is denoted by ${}^nP_r$. Here, it is to be noted that in finding the total no. of permutations of n different things taken r at a time, we consider two aspects, the selection of r things out of n together with the relative position of the r things among themselves. In the later part of this chapter, we will come across selections in which relative positions are not considered.

6.3. Factorial Notation :

The product of first n natural numbers is denoted by $n!$ or $\underline{\lfloor n}$ and is read as 'factorial n'. Thus,

$$n! = 1\times2\times3\times \ldots . \times(n-1)\times n$$

In particular,

$$1!=1$$
$$2!=1\times2=2$$
$$3!=1\times2\times3=6$$
$$4!=1\times2\times3\times4=24 \text{ etc.}$$

We can also write $n!=n\times(n-1)\times(n-2)\times \ldots . \times2\times1$

$$\Rightarrow \quad n! = n\times[(n-1)\times(n-2)\times \ldots . \times2\times1]$$
$$= n\times(n-1)!$$

Thus, $n!=n\times(n-1)! \ldots .$

Similarly, $n!=n(n-1)\times(n-2)!$

$$=n(n-1)(n-2)\times(n-3)!$$

Example 1. Evaluate (i) 6! (ii) $\dfrac{5!}{3!\times2!}$

Solution : (i) $6!=1\times2\times3\times4\times5\times6=720$

(ii) $\dfrac{5!}{3!\times2!}=\dfrac{5\times4\times3!}{3!\times2!}=\dfrac{5\times4}{2}=10.$

Example 2. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, find x.

Solution : We have, $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$

$$\Rightarrow \frac{1}{6!}+\frac{1}{7\times 6!}=\frac{x}{8\times 7\times 6!}$$

$$\Rightarrow 1+\frac{1}{7}=\frac{x}{8\times 7}$$

$$\Rightarrow \frac{8}{7}=\frac{x}{8\times 7}$$

$$\therefore \quad x=64.$$

6.4. Permutations of things all different :

Theorem 6·1. The number of permutations of n different things taking $r(\leq n)$ of them at a time is given by

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

Proof : To find the total number of permutations of n different things taken r at a time is the same as to find the total number of ways of fillings in r vacant places, say 1st, 2nd, ..., r^{th} place by the n things. The 1st place can be filled in n ways by any one of the n things. Once an element has filled in the 1st place, there are $(n-1)$ elements left. The 2nd place can be filled in by any one of these $(n-1)$ elements i.e. in $(n-1)$ ways. By in similar arguments, the 3rd place can be filled in $(n-2)$ ways, 4th in $(n-3)$ ways, ..., r^{th} in $n-(r-1)$ i.e. $(n-r+1)$ ways. So, by the fundamental counting principle, the r places can be filled in $n(n-1)(n-2)\ \ldots\ (n-r+1)$ ways.

$$\therefore \quad ^{n}P_{r}=n(n-1)(n-2)\ \ldots\ (n-r+1)$$

To make it more elegant, using the factorial notation we write $^{n}P_{r}$ as follows,

$$^{n}P_{r}=\frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 2\cdot 1}{1\cdot 2\cdot 3\cdot\ldots(n-r)}$$

$$=\frac{n!}{(n-r)!}$$

Hence, $$^{n}P_{r}=\frac{n!}{(n-r)!}$$

Note : Had the repetition of the objects any number of times been allowed, then the number of permutations of n things taken r at a time would have been $n \times n \times \ldots . r$ times i.e. n^r.

Interpretation of O!

We know that ${}^nP_r = n(n-1)\ldots.(n-r+1), \ o<r \le n.$

In particular, ${}^nP_n = n(n-1)\ldots.2 \cdot 1 = n!$

But, by the formula ${}^nP_r = \dfrac{n!}{(n-r)!}, \ o<r\le n,$

We have, $${}^nP_n = \frac{n!}{(n-n)!}$$

$$\Rightarrow n! = \frac{n!}{o!}$$

$$\Rightarrow o! = \frac{n!}{n!}.$$

From the definition of factorial, o! is not meaningful. However, in the above relation, the right side i.e. $\dfrac{n!}{n!}$ is meaningful for every +ve integer n. As such o! is interpreted as 1.

$$\therefore \ o! = 1.$$

Again, from the definition of factorial, factorial of a negative integer is meaningless.

However, $n! = n \times (n-1)!$

$$\text{i.e., } \frac{1}{(n-1)!} = \frac{n}{n!}.$$

When n is o, right side is $\dfrac{o}{o!} = \dfrac{o}{1} = o$ which is meaningful. Therefore, we have to interprete $\dfrac{1}{(-1)!}$ as 0. Similarly, we interprete $\dfrac{1}{(-m)!}$ where m is a +ve integer as o, even though $(-m)!$ is meaningless.

Example 1. Find n if ${}^{n-1}P_4 : {}^nP_5 = 1:15$

Solution : Here $^{n-1}P_4 : {}^{n}P_5 = 1:15$

$$\therefore \quad \frac{^{n-1}P_4}{^{n-1}P_5} = \frac{1}{15}$$

$$\Rightarrow \quad \frac{\dfrac{(n-1)!}{(n-1-4)!}}{\dfrac{n!}{(n-5)!}} = \frac{1}{15}$$

$$\Rightarrow \quad \frac{\dfrac{(n-1)!}{(n-5)!}}{\dfrac{n!}{(n-5)!}} = \frac{1}{15}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{15}$$

$$\therefore \quad n = 15.$$

Example 2. How many 3-digit numbers can be formed using the digits 1 to 9 if no digit is repeated ?

Solution : Here, 123 and 132 are two different numbers. So, the order in which the digits occur matters. Therefore, the required 3-digit numbers = $^{9}P_3 = \frac{9!}{(9-3)!}$ $= \frac{9!}{6!} = 7\times 8\times 9 = 504.$

Example 3. How many permutations can be formed from the letters of the word IMPHAL ?

Solution : Here *n=6, r*=6.

$\therefore$ the required number of permutations = $^{6}P_6 = 6! = 720.$

Example 4. How many 4-digit numbers can be formed using the digits 1 to 9 if repetition of the digits is allowed ?

Solution : The number of ways in which 1000's place can be filled = 9
The number of ways in which 100's place can be filled = 9
The number of ways in which 10's place can be filled = 9
and The number of ways in which units place can be filled = 9

$\therefore$ total number of 4-digit numbers that can be formed with repitition $= 9\times 9\times 9\times 9 = 9^4.$

6.5. Permutation of things not all different :

Theorem 6·2. The number of permutations of n things, where p things are of one kind, q things are of second kind and the rest are all different is $\frac{n!}{p!q!}$.

Proof : Let the total number of permutations be x. Consider one of them, say P. In P, there are p things of one kind. If we replace these by p new and different ones and permute them among themselves in all possible ways without changing the position of the others, from the single permutation P, we obtain $p!$ distinct permutations. If the same replacement and permutation is done in each of the x permutations, we will obtain altogether $p!\times x$ permutations.

Again, in each of these $p!\times x$ permutations, if the q things of second kind are replaced by q new and different things and permuted among themselves in all the possible ways, we will obtain $p!\times q!\times x$ permutations. Thus, we have $p!\times q!\times x$ permutations of n different things (since all the things of same kind have been gradually replaced by different ones). But the total number of permutations of n distinct things taken all at a time is $n!$

$$\therefore \quad n! = p!\times q!\times x$$

Hence, $$x = \frac{n!}{p!q!}$$

Note : In fact, the above result can be extended to the cases when more alike things occur in a given number of things. For this, we have a general theorem as follows.

The number of permutations of n things, where p_1 things are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of all different is $\frac{n!}{p_1!\,p_2!\,\ldots\,p_k!}$.

Example 1. Find the number of permutations of the letters of the word MATHEMATICS.

Solution : Here, there are 11 things (letters) of which there are 2A's, 2M's, 2T's and rest are all different.

$$\therefore \quad \text{the required no. of permutations} = \frac{11!}{2!\,2!\,2!}.$$

Example 2. Find the number of arrangements with the letters in the word STRANGE such that all the vowels are together.

Solution : There are 7 letters in the word STRANGE of which two are vowels and remaining five are consonants. Considering the two vowels together as a single letter different from the remaining five, we have 6 letters which can be permutted in 6!=720 ways. But in each case the vowels can be arranged among themselves in 2!=2 ways.

Hence, the numbers of arrangements with all vowels together = 720×2 = 1440.

6.6. Combinations :

In common parlance, though combination and permutation are all arrangements but in Mathematics, Combinations has a different meaning from permutations. To illustrate this, let us assume that there is a group of 3 persons namely X,Y,Z. A committee of 2 persons out of them is to be formed. In how many ways can we do so ? We know that the committee of X and Y is the same as the committee of Y and X i.e., the order in which the persons are selected, is irrelevant. In fact, there are 3 possible ways in which the committee could be formed and these are XY, YZ, ZX. Each one of these committees is an example of a combination of the 3 different things taken 2 at a time.

Suppose there are n different things. The different groups that can be formed by taking some or all of them at a time are called their combinations.

The combinations of 3 letters *a, b, c*

(1) taken all at a time is *abc.*

(2) taken two at a time are *ab, bc, ca.*

(3) taken one at a time are *a, b, c.*

Again, let us consider the set $S=\{a,b,c\}$. The subsets of S containing exactly 2 elements are $\{a,b\}$, $\{b,c\}$, $\{c,a\}$. Thus, the total number of subsets of S containing exactly 2 elements is 3. Each subset is combination of 3 elements of S taken 2 at a time. We can observe that the order in which the elements of these subsets are written is immaterial. For example, $\{a,b\}$ and $\{b,a\}$ are the same set. This makes combination different from permutation, where changing the order of the elements produces a change of permutation.

Definition : Let S be a set containing n elements and let $o \le r \le n$. Then, every subset of S containing exactly r elements is called a *combination* of the n elements taken r at a time.

The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or $\binom{n}{r}$. From the above discussions, we see that ${}^3C_3=1$, ${}^3C_2=3$, ${}^3C_1=3$.

Note : It is customary to take ${}^nC_0=1$ for any +ve integer n. In a combination, the order in which a thing occurs is immaterial. For instance, combinations *abc, acb, bca, bac, cab, cba* are all same.

The following theorem gives the relation between permutation and combination.

Theorem 6·3. $^nP_r = {^nC_r} \times r!,\ o < r \le n.$

Proof : Since, in each combination of ^nC_r, the r things can be arranged among themselves in ^rP_r i.e. $r!$ ways. So, for each of the ^nC_r combinations, there are $r!$ permutations. Thus, ^nC_r combinations have altogether $^nC_r \times r!$ permutations. But the total number of permutations of n different things taken r at a time is ^nP_r.

$$\therefore \quad {^nP_r} = {^nC_r} \times r!$$

Hence, $$^nC_r = \frac{n!}{r!(n-r)!}$$

Remarks 1. We have $^nC_r = \frac{n!}{r!(n-r)!}$

$$\therefore \quad {^nC_{n-r}} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {^nC_r}.$$

2. Taking $r = n$ in $^nC_r = \frac{n!}{r!(n-r)!}$, we have,

$$^nC_n = \frac{n!}{n!o!} = 1.$$

3. Since, $\frac{n!}{o!(n-o)!} = 1 = {^nC_o}$, the formula

$^nC_r = \frac{n!}{r!(n-r)!}$ is also applicable for $r = o$.

Theorem : $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$.

Proof : Here, $$^nC_r + {^nC_{r-1}} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-(r-1))!}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-(r-1))!}$$

$$= \frac{n![(n-r+1)+r]}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r .$$

Example 1. If ${}^nC_6 = {}^nC_7$, find ${}^nC_3$.

Solution : We have ${}^nC_6 = {}^nC_7$

$$\Rightarrow \quad \frac{n!}{6!(n-6)!} = \frac{n!}{7!(n-7)!}$$

$$\Rightarrow \quad \frac{1}{n-6} = \frac{1}{7}$$

$$\Rightarrow \quad n-6=7$$

$$\therefore \quad n=13.$$

Hence, $${}^nC_3 = {}^{13}C_3 = \frac{13!}{3!(13-3)!} = \frac{13!}{3!10!} = \frac{13\times12\times11}{3!} = 286.$$

Example 2. A bag contains 7 black and 8 red balls. Determine the number of ways in which 3 black and 4 red balls can be selected.

Solution : Here, the order in which the balls are to be selected is irrelavant. So, we have to count the number of combinations.

Now, 3 black balls can be selected from the 7 in ${}^7C_3$ ways.

Again, 4 red balls can be selected from the 8 in ${}^8C_4$ ways.

$\therefore$ total number of selections $= {}^7C_3 \times {}^8C_4$

$$= \frac{7\times6\times5}{1\times2\times3} \times \frac{8\times7\times6\times5}{1\times2\times3\times4}$$

$$= 2450.$$

EXERCISE 6.1

1. Evaluate

   (i) $7!$ (ii) $6!-5!$.

2. Compute $\frac{9!}{6!\times 3!}$.

3. If $\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$, find x.

4. Evaluate ${}^{n}P_{r}$ when

   (i) $n=5,\ r=3$ (ii) $n=8,\ r=4$.

5. Find n if ${}^{n-1}P_{5}:{}^{n}P_{6}=1:11$.

6. Find r if (i) ${}^{5}P_{r}=2\,{}^{6}P_{r-1}$ (ii) ${}^{5}P_{r}={}^{6}P_{r-1}$.

7. If ${}^{n}C_{8}={}^{n}C_{2}$, find ${}^{n}C_{3}$.

8. Determine n if

   (i) ${}^{2n}C_{4}:{}^{n}C_{4}=42:1$ (ii) ${}^{2n}C_{3}:{}^{n}C_{3}=11:1$.

9. Prove that (i) ${}^{n}C_{r}+2\,{}^{n}C_{r-1}+{}^{n}C_{r-2}={}^{n+2}C_{r}$

   (ii) ${}^{n}C_{r}+{}^{n-1}C_{r}+{}^{n-2}C_{r}+\ldots+{}^{r}C_{r}={}^{n+1}C_{r+1}$.

10. Prove that $\frac{{}^{4n}C_{n}}{{}^{2n}C_{n}}=\frac{(4n-1)(4n-3)\ldots 5\cdot 3\cdot 1}{\{(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1\}^{2}}$.

11. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated ?

12. How many 6-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 24 and no digit appears more than once ?

13. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once ?

14. In how many ways can the letters of the word PERMUTATIONS be arranged if
    (i) the words start with P and end with S
    (ii) the vowels are all together
    (iii) there are always 4 letters between P and S ?
15. How many chords can be drawn through 25 points on a circle ?
16. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in the combination.
17. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl, if each cricket team of 11 must include exactly 4 bowlers ?
18. Two political parties having respectively 15 and 23 MLA's form a government. In how many ways can a cabinet with 7 ministers holding different portfolios be formed such that 3 are from the party with fewer number of MLA's.
19. A photo of ten students, 5 Indians, 3 Americans and 5 Chinese is to be taken so that all the Indians are together and so are the Americans and the Chinese. How many photographs are possible ?
20. In how many ways can 10 boys and 7 girls be arranged in a row so that no two girls are together ?
21. Numbers are formed by using all the digits 2, 5, 6, 8, 9. How many of the numbers so formed are divisible by 2 or 5 ? (Digits not repeated)
22. A row of 5 chairs is to be occupied by 3 boys and 2 girls. There are 7 boys and 4 girls present. In how many ways can the line be formed, if the two end chairs have to be occupied by 2 boys ?
23. A rectangle is cut by 6 lines parallel and 8 lines perpendicular to the base. Find the total number of rectangles thus formed.
24. From 10 teachers and 4 students, a committee of 6 is to be formed. In how many ways can this be done (a) when the committee contains exactly 2 students, (b) atleast 2 students.

ANSWERS

1. (i) 5040 (ii) 600 2. 84 3. 121 4. (i) 60 (ii) 1680 5. 11

6. (i) 3. (ii) 4. 7. 120 8. (i) $n = 5$ (ii) $n = 6$ 11. 108 12. 1680 13. 8!

14. (i) $\frac{10!}{2!}$ (ii) $\frac{8!}{2!} \times 5!$ (iii) 25401600 15. 300 16. 778320 1

18. ${}^{23}C_4 \times {}^{15}C_3 \times 7!$ 19. $5! \times 3! \times 2! \times 3!$ 20. $10! \times {}^{11}P_7$ 21. $4 \times 4!$ 22. 210×36

23. 1260 24. (a) ${}^{4}C_2 \times {}^{10}C_4$ (b) 1785.

Chapter–7

VECTORS

7.1 Introduction

In our common day-to-day life or in systematic scientific investigations, particularly in physical sciences, we come across many entities which can be completely determined by the measures of specific quantities they contain. For example, take the volume of a body. The volume of the body is the amount of space it contains and it is quantified in units like cubic centimeter, litre etc. Similarly, the areas of an enclosure is the amount of surface that the enclosure contains and it is quantified in suitable units like square cm, hectare etc. In any case, the measures are the magnitudes of the entities. Such entities which can be determined completely when only their magnitudes are known are called *scalar quantities* or *simply scalars*. Some common example of scalars are volume, area, length, density, mass etc.

Now imagine the term 'direction', more precisely, the direction of the location of a point. It is the indication of the position of the point relative to some standard point. Generally, the standard point is the permanent point in the celestial sphere towards which the Earth's axis is pointing and its location is identified by the pole star. This point is called the *north celestial pole.* The direction of the location of the particular point is fixed by quantification of its deviation from the standard point in suitable units like degree, minute etc. In fact, direction is an attribute measured by quantification. Later on you will come across terms like direction cosines and direction number etc.

Now-a-days many attributes are quantified in suitable units. For example, the attribute of beauty is measured in units called *helen, millihelen* etc. These units are in honour of the legendary beauty queen, Helen of Troy. Similarly, hardness of a substance is measured in a scale called *Moh's scale* of hardness.

An entity which can be completely determined only when its magnitude and direction are known is called *a vector quantity* or simply *a vector*. In other words, an entity having both magnitude and direction is called *a vector*. Some common examples of vectors are *velocity, acceleration, force* etc.

There are physical quantities which cannot be completely known even if their magnitudes and directions are known. For example, to know stress, we have to know the area over and above a force. Such entities are not vectors. They are tensors and they will not be discussed in this book.

7.2 Symbols representing vectors

Since a vector has both magnitude and direction, symbols like a, b, c, x, y, z,..... etc. which represent scalars are not suitable to represent vectors. In printed materials clarendon or thick letters like ***a*, *b*, *c*,** ....etc. are used to denote vectors. However, in veiws of the difficulties in differentiating thick from thin letters in manuscripts, symbols like $\vec{a}, \vec{b}, \vec{c}$ ....etc. with arrows overhead are used to denote vectors while *a*, *b*, *c*,...etc. denote scalars representing their respective magnitudes. We also use $|\vec{a}|, |\vec{b}|, |\vec{c}|$.... etc. to denote the respective magnitudes of $\vec{a}, \vec{b}, \vec{c}$ ....etc.

7.3 Representation of vectors

We know that a directed line segment has both length and direction. The length being a scalar, together with its direction, a directed line segment is similar in nature to a vector. Thus, if a vector and a line segment are equally directed, then by taking the length of the line segment proportional to the magnitude of the vector, it can represent a vector.

In Fig 7.1, $\overrightarrow{AB}$ is a directed line segment.

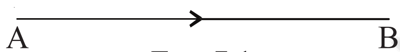


Fig. 7.1

If a vector $\vec{a}$ and $\overrightarrow{AB}$ are equally directed and $|\vec{a}|$, the magnitude of $\vec{a}$ is taken equal to AB, the length of $\overrightarrow{AB}$ in some scale, then, we write $\vec{a} = \overrightarrow{AB}$. Thus, $\overrightarrow{AB}$ is taken as a vector.

For the vector $\overrightarrow{AB}$, A and B are respectively called the *initial and terminal points of the vector.*

7.4 Types of vectors

Zero vector : A vector whose magnitude is 0 (zero) is called *a zero* or *a null vector*. The direction of a zero vector is any arbitrary direction. For a zero vector, the initial and the terminal points coincide.

Proper vector : Any vector other than a zero vector is called *a proper vector*.

Co-initial vectors : Vectors are said to be co-initial if they have the same initial point.

Co-terminus vectors : Vectors are said to be co-terminus if they have the same terminal point.

Equal vectors : Vectors are said to be equal if they have the same magnitude and are equally directed.

Like vectors : Vectors are said to be like if they are equally directed.

Unlike vectors : Vectors are said to be unlike if they are parallel but their directions differ by 180°.

Negative of a vector : A vector equal in magnitude to a vector $\vec{a}$ but just opposite to the direction of $\vec{a}$ is called the *negative of* $\vec{a}$. It is denoted by $-\vec{a}$. Surely, $|-\vec{a}| = |\vec{a}|$.

Scalar Multiple of a vector : For a proper vector $\vec{a}$ and a scalar $m(\neq 0)$, $m\vec{a}$ is a vector whose magnitude is $|m|$ times $|\vec{a}|$ but having the same or opposite direction as that of $\vec{a}$ according as $m>0$ or $m<0$. It is multiple or a submultiple of $\vec{a}$ according as $|m|$ is more than one or less than one.

Unit vector : A vector whose magnitude is one is called *a unit vector*. For any proper vector $\vec{a}$, $\dfrac{\vec{a}}{|\vec{a}|}$ is a unit vector in the direction of $\vec{a}$. Usually we use the symbol $\hat{a}$ (a cap) to denote a unit vector in the direction of $\vec{a}$. Thus, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ or $\dfrac{\vec{a}}{a}$, so that, $a\ \hat{a} = \vec{a}$.

Reciprocal of a vector : For any proper vector $\vec{a}$, the vector having the same direction as that of $\vec{a}$ but whose magnitude is the reciprocal of the magnitude of $\vec{a}$ is called the *reciprocal vector of* $\vec{a}$.

The vector $\dfrac{\vec{a}}{a^2}$ is the reciprocal of $\vec{a}$.

Note that $\left|\dfrac{\vec{a}}{a^2}\right| = \dfrac{a}{a^2} = \dfrac{1}{a}$

Free and localized vectors : A vector identified by its magnitude and direction but not having any specific position in space is called *a free vector*. On the other hand, a vector having a specific position in space is called *a localized vector*.

Two free vectors are equal when they have the same magnitude and direction whereas two localized vectors are equal only when they occupy identical positions in space over and above having the same magnitude and direction.

Position vector (of a point) : The vector identifying the position of a point P in space with reference to a particular point is called the *position vector* of the point P. In general, we take the origin O in the system of cartesian co-ordinates as the point of reference.

Thus, the vector $\vec{r}$ given by $\vec{r} = \overrightarrow{OP}$ is the position vector of P. We also use the symbol $P(\vec{r})$ to denote the position vector of P.

7.5 Composition of vectors

Since vectors like $\vec{a}, \vec{b}, \vec{c}$ ....etc. are mathematical entities, we are to examine the validity of the basic fundamental operations of addition, subtraction, multiplication and division and other properties among vectors.

In this book, we shall consider the operations of addition and subtraction of vectors and some other properties including multiplication of a vector by a scalar.

Addition of two vectors

Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{LM}$ be two vectors as in Fig. 7.2

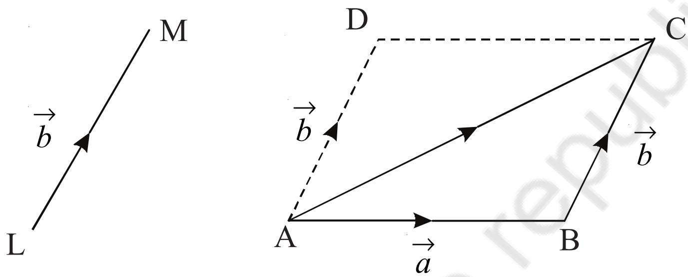


Fig. 7.2

To find the sum of $\vec{a}$ and $\vec{b}$, we draw $\overrightarrow{BC} = \overrightarrow{LM} = \vec{b}$ such that the initial point of $\overrightarrow{BC}$ is the terminal point of $\overrightarrow{AB}$. If $\overrightarrow{AC} = \vec{c}$ then, it is defined as the sum of $\vec{a}$ and $\vec{b}$ and we write

$$\vec{a} + \vec{b} = \vec{c}$$

$$\text{or } \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

This is sometimes referred to as the triangle law of addition.

If we draw $\overrightarrow{AD} = \overrightarrow{LM}$, then we see that AC is the diagonal of the parallelogram ABCD with AB and AD as two adjacent sides.

Thus, to find the sum of any two arbitrary vectors $\vec{r}$ and $\vec{s}$, draw a parallelogram with two adjacent sides representing $\vec{r}$ and $\vec{s}$ so that they are co-initial, as in Fig. 7.3.

Then, the diagonal of the parallelogram drawn from the common initial point represents the sum of $\vec{r}$ and $\vec{s}$ i.e. $\vec{r} + \vec{s}$.

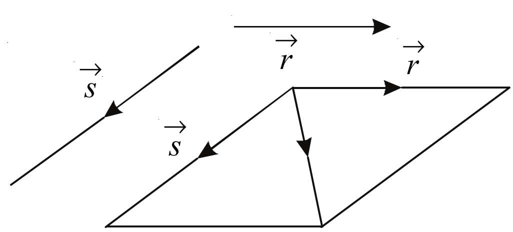


Fig. 7.3

From this consideration, we can now define vectors in a more definitive way as entities having both magnitude and direction and also obeying the parallelogram law of addition.

Deductions

(i) From the triangle inequalities, we know that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. As vector addition follows the triangle law of addition, the magnitude of the vectors obey the triangle inequality.

In the triangle ABC, Fig. 7.4

Take $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$ and $\vec{c} = \overrightarrow{AC}$,

then, from $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

we get $\vec{c} = \vec{a} + \vec{b}$

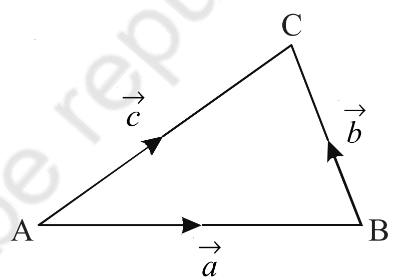


Fig. 7.4

But from the triangle inequalities, we get

$$AC < AB + BC$$

i.e. $|\overrightarrow{AC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}|$

i.e. $|\overrightarrow{AB} + \overrightarrow{BC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}|$

or $|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$

In short, if $\vec{c} = \vec{a} + \vec{b}$, we get

$$|\vec{c}| < |\vec{a}| + |\vec{b}|$$

or $|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$

However, if $\vec{a}$ and $\vec{b}$ are like vector, we get

$$|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$$

Thus, we combine the two results into one as

$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$, where the equality holds when $\vec{a}$ and $\vec{b}$ are like vectors.

(ii) We have seen how to find the sum of any two arbitrary vectors say $\vec{a}$ and $\vec{b}$.

If we take the negative of $\vec{a}$ i.e. $-\vec{a}$ in place of $\vec{b}$ in ($\vec{a} + \vec{b}$), we shall find the sum of $\vec{a}$ and $-\vec{a}$.

In Fig 7.5, $\vec{a} = \overrightarrow{OA}$ so that its initial point is O and terminal point is A.

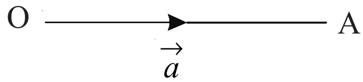


Fig. 7.5

By the definition of the negative of a vector, $-\vec{a} = \overrightarrow{AO}$. Its initial point is A and terminal point is O. Thus, when we add $\vec{a}$ and $-\vec{a}$ we take the terminal point of $\vec{a}$ i.e. A as the initial point of $-\vec{a}$. Surely, the terminal point of $-\vec{a}$ coincides with the initial point of $\vec{a}$ i.e. the point O.

Therefore, the net effect of $\vec{a} + (-\vec{a})$ is given by $\vec{a} + (-\vec{a}) = \overrightarrow{OA} + \overrightarrow{AO} = \overrightarrow{OO}$ (from point O to itself)

$$\Rightarrow \vec{a} + (-\vec{a}) = \vec{O}, \text{ zero vector.}$$

(iii) Considering the parallelogram ABCD, Fig. 7.6, from the definition of the sum of two vectors, we see that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Also, $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$

But $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{DC} = \overrightarrow{AB}$

$\therefore$ $\overrightarrow{BC} + \overrightarrow{AB} = \overrightarrow{AC}$

Thus, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}$

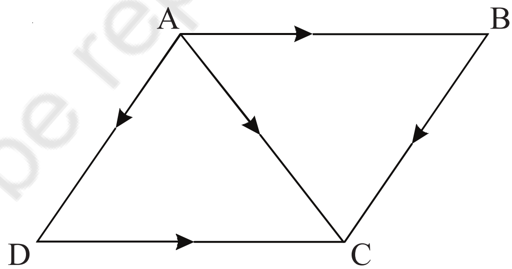


Fig. 7.6

In short, for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ i.e. vector addition is commutative.

(iv) Consider the quadrilateral ABCD in which $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{CD} = \vec{c}$ in Fig 7.7.

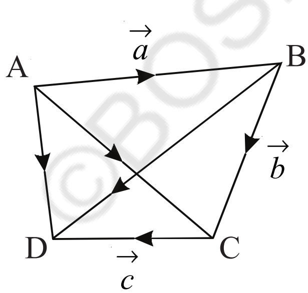


Fig. 7.7

Now, $(\vec{a} + \vec{b}) + \vec{c} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD}$

$= \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

Again, $\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD})$

$= \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

Thus, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ i.e. vector addition is associative.

7.6 Subtraction of Vectors

While dealing with real numbers, we have seen that the operation of subtraction is the reverse process of the operation of addition. Also, it has been seen that corresponding to an addition fact there are two subtraction facts. We shall show that the same are also true in the case of vectors.

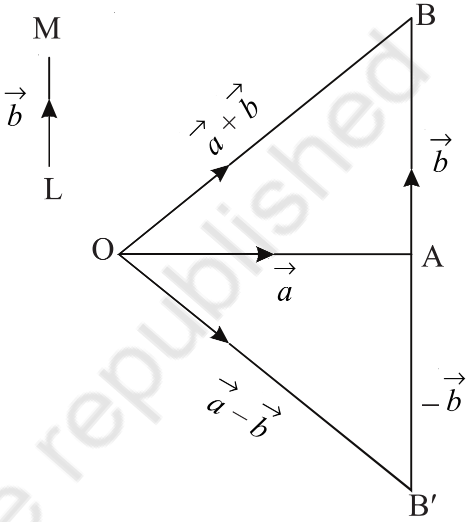


Fig. 7.8

In Fig. 7.8, $\overrightarrow{OA}$ is the vector $\vec{a}$ and $\overrightarrow{LM}$ is vector $\vec{b}$.

To find $\vec{a}+\vec{b}$, we draw $\overrightarrow{AB}=\overrightarrow{LM}$ (From the terminal point of $\overrightarrow{OA}$) so that $\overrightarrow{AB}=\vec{b}$

Now, $\overrightarrow{OB}=\vec{a}+\vec{b}$

To find $\vec{a}+(-\vec{b})$ i.e. $\vec{a}-\vec{b}$, we draw $\overrightarrow{AB'}=-\overrightarrow{LM}$, reversing the direction of $\vec{b}$ but maintaining its magnitude, so that $\overrightarrow{AB'}=-\vec{b}$

Then, $\overrightarrow{OB'}=\vec{a}-\vec{b}$.

If $\overrightarrow{OB}=\vec{c}$, then $\vec{c}=\vec{a}+\vec{b}$ .... (i)

Adding $-\vec{b}$ on both sides of (i), we get

$$\vec{c}-\vec{b}=\vec{a}+\vec{b}-\vec{b}$$

$$\Rightarrow \quad \vec{c}-\vec{b}=\vec{a}+(\vec{b}-\vec{b})=\vec{a}+\vec{0}$$

$\Rightarrow \quad \vec{c}-\vec{b}=\vec{a}$ .... (ii)

Similary, $\vec{c}-\vec{a}=\vec{b}$ .... (iii)

Thus, corresponding to one addition fact (i) we get two subtraction facts (ii) and (iii).

These results show that like algebraic quantities vectors can be transported across the sign of equality. In a chemical equation transportation across the sign of equality is not possible e.g. in the equation $2H + O = H_2O$, a transportation like $2H = H_2O - O$ is meaningless.

Observation :

In Fig. 7.9, if we complete the two parallelograms OABC and OAB′C′ as shown in Fig. 7.9, the two vectors $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ are $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ respectively. Noting that AC=OB′, we see that the magnitudes of $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are the length of the diagonals of the parallelogram OABC.

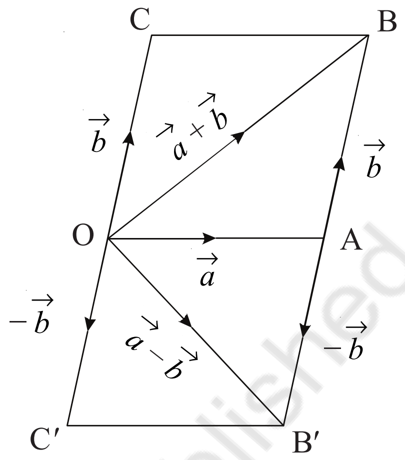


Fig. 7.9

Corollary :

If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then,

$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

This agrees with the fact that the magnitudes of $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are the lengths of the diagonals of a rectangle which are equal in length.

7.7 Scalar multiplication and distributive law of scalar multiplication

We have stated earlier that the product of a scalar and a vector is defined. For a scalar m>O and a vector $\vec{a}$ their product in the vector $m\vec{a}$ whose magnitude is m times the magnitude of $\vec{a}$ and is in the direction of that of $\vec{a}$

We shall now show that the scalar multiplication of the sum of the vectors is distributive over vector addition. Symbolically,

$m(\vec{a}+\vec{b}) = m\vec{a}+m\vec{b}$.

In Fig, 7.10, $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{AB}=\vec{b}$ and accordingly $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$ i.e. $\overrightarrow{OB}=\vec{a}+\vec{b}$. Produce OA to a point C such that OC = mOA. Through C, draw a line parallel to AB meeting OB at the point D, when produced.

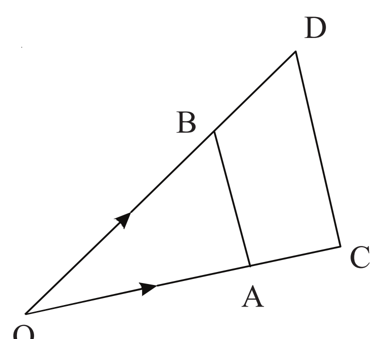


Fig. 7.10

Now, from the similar triangles OAB and OCD,

$$\frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA} = m$$

$$\Rightarrow OD = m\,OB,\ CD = m\,AB,\ OC = m\,OA$$

Again, $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$$\Rightarrow m\,\overrightarrow{OB} = m\,\overrightarrow{OA} + m\,\overrightarrow{AB}$$

$$\Rightarrow m\,(\vec{a} + \vec{b}) = m\,\vec{a} + m\,\vec{b}$$

Similarly, we can prove that $(m+n)\,\vec{a} = m\,\vec{a} + m\,\vec{a}$

7.8 To find the position vector of a point dividing internally in a ratio, the join of two points of given position vectors.

In Fig. 7.11, A and B are the two given points whose position vectors with reference to the origin O are $\vec{a}$ and $\vec{b}$ respectively i.e. $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. C is the point dividing AB internally in the ratio $m : n$.

We are to find the position vector of C say, $\vec{c}$ i.e. we are to find $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.

Now, $\frac{AC}{CB} = \frac{m}{n} \Rightarrow AC = \frac{m}{n} \cdot CB$.

This gives us, $\overrightarrow{AC} = \frac{m}{n}\overrightarrow{CB}$

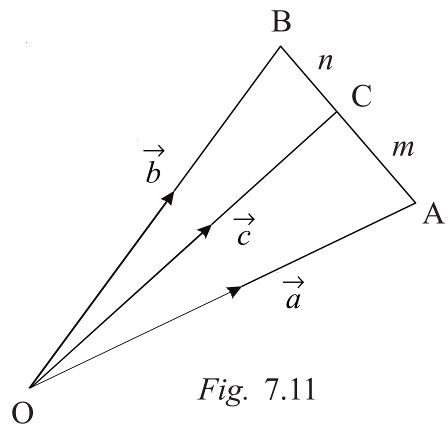


Fig. 7.11

$$\Rightarrow n\,(\overrightarrow{AO} + \overrightarrow{OC}) = m\,(\overrightarrow{CO} + \overrightarrow{OB})$$

$$\Rightarrow n\,(-\overrightarrow{OA} + \overrightarrow{OC}) = m\,(-\overrightarrow{OC} + \overrightarrow{OB})$$

$$\Rightarrow n\,(-\vec{a} + \vec{c}) = m\,(-\vec{c} + \vec{b})$$

$$\Rightarrow m\,\vec{c} + n\,\vec{c} = n\,\vec{a} + m\,\vec{b}$$

$$\Rightarrow (m+n)\,\vec{c} = n\,\vec{a} + m\,\vec{b}$$

$$\Rightarrow \vec{c} = \frac{n\,\vec{a} + m\,\vec{b}}{m+n}$$

Deduction : If C is the mid point of AB, then its position vector is $\frac{\vec{a} + \vec{b}}{2}$.

Work out examples :

Example 1 : Show that the three medians of a triangle are concurrent.

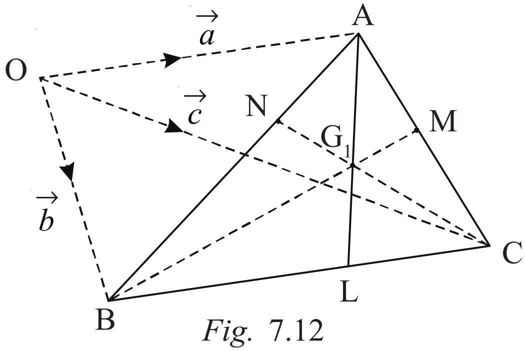


Fig. 7.12

Solution : Let ABC be a triangle. With reference to an arbitrarily chosen point O as orgin let the position vectors of the vertices A, B and C be respectively $\vec{a}$, $\vec{b}$ and $\vec{c}$.

i. $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.

If L, M and N are the mid points of the sides BC, CA and AB respectively, then their position vectors are given by

$$\overrightarrow{OL} = \frac{\vec{b} + \vec{c}}{2}, \quad \overrightarrow{OM} = \frac{\vec{a} + \vec{c}}{2} \text{ and } \overrightarrow{ON} = \frac{\vec{a} + \vec{b}}{2}$$

Let G_1 be the point dividing the median AL in the ratio 2 : 1, then its position vector $\overrightarrow{OG_1}$ is given by

$$\overrightarrow{OG_1} = \frac{2 \cdot \overrightarrow{OL} + 1 \cdot \overrightarrow{OA}}{2+1}$$

$$= \frac{2 \times \frac{\vec{b} + \vec{c}}{2} + \vec{a}}{3}$$

i.e. $$\overrightarrow{OG_1} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

Again, if G_2 be the point dividing the medium BM in the ratio 2 : 1 then, its position vector $\overrightarrow{OG_2}$ is given by

$$\overrightarrow{OG_2} = \frac{2 \cdot \overrightarrow{OM} + 1 \cdot \overrightarrow{OB}}{2+1}$$

$$= \frac{2 \times \frac{\vec{a} + \vec{c}}{2} + \vec{b}}{3}$$

i.e. $\vec{OG_2} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

Similarly, if G_3 be the point dividing the median CN in the ratio 2 : 1, then we shall get

$$\vec{OG_3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

Now, the three points G_1, G_2, G_3 have the same position vector referred to the same origin. This is possible only when the three points coincide at the same point say, G.

This point G is called the centroid of the triangle ABC and it is a point of trisection of each of the three medians.

Example 2 : Show that the two diagonals of a parallelogram bisect each other.

Solution : Let ABCD be a parallelogram.

With reference to an arbitrary origin O, let the position vectors of the four vertices

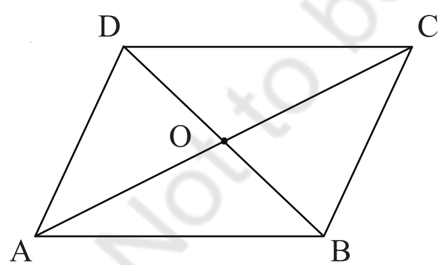


Fig. 7.13

A, B, C and D be respectively $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$.

Now, the two sides AB and DC are equal and parallel.

i.e. vectorially, $\vec{AB} = \vec{DC}$

$\Rightarrow \vec{OB} - \vec{OA} = \vec{OC} - \vec{OD}$

$\Rightarrow \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \vec{a} + \vec{c} = \vec{b} + \vec{d}$ .... (i)

Now, the position vector of the mid point of the diagonal AC is $\frac{\vec{a} + \vec{c}}{2}$.

Also, the position vector of the mid point of the diagonal BD is $\frac{\vec{b}+\vec{d}}{2}=\frac{\vec{a}+\vec{c}}{2}$ by (i).

Thus, the mid-points of the two diagonals have the same position vector. This shows that two the mid-points coincide with each other. In other words, the two diagonals bisect each other.

Example 3 : Show that the figure formed by joining the mid points of the sides of a quadrilateral taken in order is a parallelogram.

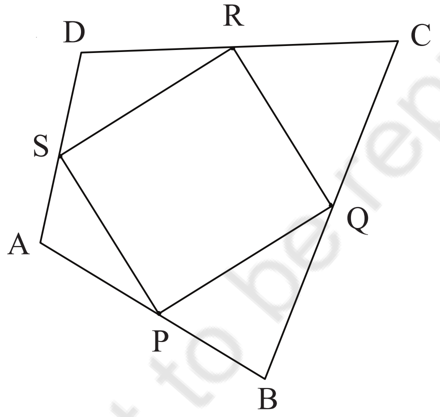


Fig. 7.14

Solution : Let ABCD be a quadrilateral. With reference to an arbitrary origin O, let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the respective position vectors of the vertices A, B, C and D.

If P, Q, R and S are the respective mid points of the four sides AB, BC, CD and DA, then their position vectors are given by

$$\overrightarrow{OP}=\frac{1}{2}(\vec{a}+\vec{b}),\ \overrightarrow{OQ}=\frac{1}{2}(\vec{b}+\vec{c}),\ \overrightarrow{OR}=\frac{1}{2}(\vec{c}+\vec{d}) \text{ and } \overrightarrow{OS}=\frac{1}{2}(\vec{a}+\vec{d})$$

Now, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\frac{1}{2}\left[(\vec{b}+\vec{c})-(\vec{a}+\vec{b})\right]=\frac{1}{2}(\vec{c}-\vec{a})$

and $\overrightarrow{SR}=\overrightarrow{OR}-\overrightarrow{OS}=\frac{1}{2}\left[(\vec{c}+\vec{d})-(\vec{a}+\vec{d})\right]=\frac{1}{2}(\vec{c}-\vec{a})$

Then, $\overrightarrow{PQ}=\overrightarrow{SR}$

This shows that the sides PQ and SR are equal and parallel. As such the figure PQRS is a parallelogram.

EXERCISE 7

1. Separate the following entities into scalars and vectors :

   Age, mass, time, length, density, pressure, displacement, velocity, force, specific gravity, electric current, temperature, momentum, weight and acceleration.

2. If $|\vec{a}| = 5$, then find the scalar m such that $|m\vec{a}| = 15$.
3. If $|\vec{a}| = 10$, find $\hat{a}$ and the raciprocal vector of $\vec{a}$.
4. In a triangle, show that the line joining the mid points of any two sides is parallel to the third side and half of its length.
5. In a triangle ABC, D, E and F are respectively the mid-points of the sides BC, CA and AB.

   For any arbitrary point P, show that

   $$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} = \overrightarrow{PD} + \overrightarrow{PE} + \overrightarrow{PF}$$

6. If $\vec{a}$ and $\vec{b}$ are the adjacent sides of a regular hexagon taken in order, find the vecotrs determined by the other sides of the hexagon taken in the same order.
7. ABCD is a parrallelogram and E is the mid-point of BC. Show that AE and BD trisect each other.
8. Show that the sum of the vectors determined by the medians of a triangle directed from the vertices is zero.

ANSWERS

1. Scalars : Age, mars, time, length, density, pressure, specific gravity, electric current and termperature.

   Vectors : Displacement, velocity, force, momentum, weight and acceleration.

2. ± 3

3. $\frac{\vec{a}}{10}, \frac{\vec{a}}{100}$

6. $\vec{b} - \vec{a}, \; -\vec{a}, \; -\vec{b}, \; \vec{a} - \vec{b}$

Chapter–8

DYNAMICS

8.1 Introduction

Mechanics is considered as the foundation of modern physical sciences. Mechanics is divided broadly into two branches, *Dynamics* which deals with bodies in motion and *Statics* which deals with bodies at rest under the action of forces.

Dynamics is the science of moving bodies. However, Dynamics of liquids and gases are dealt separately under the names Hydrodynamics or Fluid Dynamics and Dynamics of gases respectively.

But, when we say Dynamics, it is concerned with the Dynamics of non-deformable solid bodies.

Again, Dynamics is bifurcated into Dynamics of a Particle and Dynamics of Rigid bodies.

In the Dynamics of a Particle we do not consider the shape and size of the body in motion. Only the principles and the laws governing the bodies in motion are considered. In fact, in Dynamics of a Particle the motion of the body is considered on the assumption that its whole mass consisting of the mass of every part of the extended body is concentrated at its centre of mass (or centre of gravity)*.

Thus, under this stated condition the motion of the body is analogous with the motion of a point of a particle whose mass is equal to the mass of the body.

In the Dynamics of a Rigid body, the shape and size of the body in the motion are also taken into consideration.

In this book, we shall study Dynamics of a particle only.

Dynamics of a particle is generally known by the short title Dynamics which is studied under two heads viz. Kinematics and Kinetics.

8.2 Kinematics and Kinetics

In Kinematics, the motion of a body under different modes of movement in connection with time and space without considering the causes and effects of the motion is studied. Basically, Kinematics deals with the geometrical aspect of the motion of a body for which entities like mass of the body are irrelevant.

In Kinetics, the various causes and effects of the motion of a body or a system of bodies are considered. It rather deals with the physical aspect of the motion of a body where mass of the body is indispensable.

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* The centre of mass of a body is a point associated with the body where the whole mass of th body is supposed to be concentrated.

8.3 Motion and Displacement

A particle is said to be in motion when it changes its position from a point to any other point by any path either a straight line or a curve. When the particle does not change its position, it is said to be at rest. Motion of a particle will be called *rectilinear* or *curvilinear* according as its path is straight or curved.

The displacement of a moving particle in any time interval is its change of position during the interval as indicated by the straight line joining its initial and final position during the interval. If a particle moves from a point A to a point B, along any path, then its displacement is the straight length $\overrightarrow{AB}$ in that direction. Thus, displacement of a particle moving from A to B possesses three fundamental characteristics :

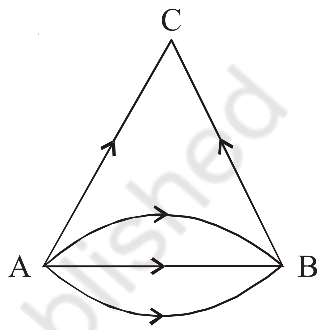


Fig. 8.1

(i) magnitude – represented by the length AB

(ii) direction – along $\overrightarrow{AB}$.

(iii) sense – from A to B.

Displacement is therefore a vector quantity as it possesses both magnitude and direction.

8.4 Speed and Velocity

The speed of a moving particle is defined to be the rate of describing its path. Speed does not give any idea of the direction along which the particle is moving and hence it is a scalar quantity possessing only the magnitude. The speed is said to be uniform when the particle moves through equal lengths of its path in equal intervals of time, however small these equal time intervals may be. Otherwise, the speed is said to be non-uniform (or variable). The *average speed* of a moving particle for any finite interval of time is the ratio of the distance to the time taken by the particle in traversing the distance. Mathematically, if a length s of its path is described in an interval of time t, then the average speed during this interval is $\frac{s}{t}$. Thus, the average speed of a moving particle for any prescribed interval of time during its motion is that uniform speed with which the particle would describe the same length of its path in the same interval of time.

The velocity of a moving particle is defined to be the rate of its displacement i.e., the rate of change of its position. It is a vector quantity in the sense that it has magnitude and direction. Then, the velocity of a particle moving in a straight line is a vector whose magnitude is the speed of the particle and whose direction is the direction along which the particle is moving.

The velocity of a moving particle is said to be uniform, when it always moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, however small these time intervals may be. Otherwise, the velocity is said to be non-uniform (or variable). The average velocity of a moving particle during any interval of time is that uniform velocity with which the particle would describe the same displacement in the same time. Mathematically, if d is the length of the line segment joining the points A and B, the initial and final position respectively corresponding to an interval of time t, then the average velocity is $\frac{d}{t}$, in the direction along $\overrightarrow{AB}$.

Unit of velocity : While learning the unit of velocity, it is suggested that the three standard systems of units of measurements involving the three fundamental quantities namely, length, mass and time may be introduced. They are (i) F.P.S. (foot-pound-second) system or British system (ii) C.G.S. (centimeter-gram-second) system or French system and (iii) M.K.S. (metre-kilogram-second) system or International system of units or in short, SI-units.

The unit of velocity in F.P.S. system is foot per second, in C.G.S. system is centimeter per second or cm/sec and in M.K.S. system is metre per second or m/sec.

8.5 Motion with Uniform Velocity

Let us find the distance described by a particle moving a long a straight line with uniform velocity.

Let v be the uniform velocity of the particle.

The distance traversed in 1 unit of time = v.

The distance traversed in 2 units of time = $2v$.

The distance traversed in 3 units of time = $3v$ and so on.

Thus, the distance traversed in t units of time = tv.

If s denotes the total distance described in t units of time, then

$$s = vt$$

$$\boxed{\text{distance} = \text{uniform velocity} \times \text{time.}}$$

8.6 Acceleration

Acceleration of a moving particle is the rate of change of its velocity. It is also a vector quantity as it has both magnitude and direction. Acceleration of a moving particle is said to be uniform when equal change of velocity in the same direction takes place in equal intervals of time, however small these equal time-intervals may be. Otherwise, it is said to be non-uniform (or variable).

When the velocity of a moving particle increases, the acceleration is positive, and when the velocity decreases, the acceleration is negative. A negative acceleration is known as *retardation*.

The unit of acceleration is F.P.S. system is foot per second per second (ft/sec^2) in C.G.S. system in centimetre per second or cm/sec^2 and in M.K.S. system is metre per second per second or m/sec^2.

Acceleration due to gravity :–

When a heavy body is dropped from a height, it falls vertically towards the earth. It may be noticed that its velocity, which is initially zero, continually increases as it falls. In other words, during the fall, the motion is uniformly accelerated. It is due to the attraction of the earth (earth's gravitation). The acceleration produced due to this gravitation of the earth is called the *acceleration due to gravity*. It is generally denoted by "g". Its value has been determined accurately by various methods, among which mention may be made of the well-known pendulum experiments. The value of g is found to vary slightly from place to place on the surface of the earth, from 32.091 ft/sec^2 at the equator to 32.252 ft/sec^2 at the poles. For numerical examples, this value is taken as 32 ft/sec^2 in F.P.S. system ; 981 cm/sec^2 in C.G.S. system and 9.8 m/sec^2 in M.K.S. system or SI-units.

The acceleration due to gravity is discovered by Gali-Galileo towards the close of the 16th century.

8.7 Rectilinear Motion with Uniform Acceleration

Let a particle start with a velocity u and move a long a straight line with uniform acceleration f. Let v be the velocity of the particle and s be its displacement at the end of any interval of time t. Then the following formulae will be established.

(i) $v = u + ft$

(ii) $s = ut + \frac{1}{2}ft^2$

(iii) $v^2 = u^2 + 2fs$

I. To establish the formula $v = u + ft$.

Here u is the initial velocity and v is the final velocity corresponding to any interval of time t during the motion of the particle along a straight line with uniform acceleration f.

$\therefore$ the change of velocity during the interval $t = v - u$

Also, it is given that this change is at a uniform rate f.

$$\therefore \quad \frac{v-u}{t} = f$$

$$\Rightarrow \quad v - u = ft$$

$$\Rightarrow \quad v = u + ft$$

II. To establish the formula $s = ut + \frac{1}{2}ft^2$.

If the particle acquires a velocity v at the end of the interval of time t, then we have seen that

$$v = u + ft.$$

We make the observation that the velocity of the body increases from u to v at the uniform rate of f per unit time. Let us imagine that the interval t is divided into n equal sub-units of duration $\frac{t}{n}$ each.

Now, velocity of the particle at the end of one such sub-unit of time after start $= u + f \cdot \frac{t}{n}$.

Again, velocity of the particle at just one such sub-unit of time before the end of the interval $= v - f \cdot \frac{t}{n}$.

Therefore, the average of these two velocities $= \dfrac{\left(u+\frac{ft}{n}\right)+\left(v-\frac{ft}{n}\right)}{2}$

$$= \frac{1}{2}(u+v)$$

This value is independent of the number of sub-units into which t has been divided.

Then, the average velocities of the particle at instances equally gaped from the begining and the end of the interval t are the same and the common value is $\frac{1}{2}(u+v)$.

In fact, $\frac{1}{2}(u+v)$ i.e., $\frac{1}{2}(u+u+ft)$ i.e., $u+\frac{1}{2}ft$ is the velocity of the body at the end of the first half of the intervel t i.e. at the middle of the interval.

This enables us to look upon the motion of the particle as if it is travelling with a uniform velocity $\frac{1}{2}(u+v)$ throughout the interval of time t thereby covering the distance s during this time t. Because of the compensatory nature of the average value when considered over a period of time, this method entails no error in finding the total distance covered during the interval of time t throughout which the average velocity $\frac{1}{2}(u+v)$ is taken as a uniform velocity. As such,

$$\begin{aligned} s &= \frac{1}{2}(u+v)t \\ &= \frac{1}{2}(u+u+ft)t \\ &= ut+\frac{1}{2}ft^2 \end{aligned}$$

$$\text{i.e. } s = ut+\frac{1}{2}ft^2$$

Graphical Illustration :

Let us first take the case of a particle moving along a straight line with a uniform velocity v.

If we make a table of values of the velocity V at the end of different successive units of time T that elapsed, reckoned from the start, we get

Time : T	0	1	2	....	t
Velocity : V	v	v	v	....	v

The graph of the velocity of the particle against time is a straight line CB parallel to the time - axis at a distance of v, as shown in Fig. 8.2.

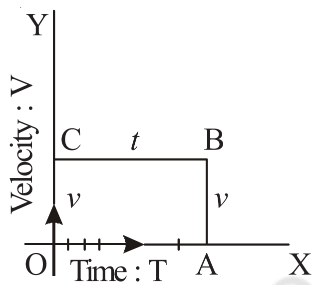


Fig. 8.2

Since $s = vt$, (when v is uniform)

$\therefore \quad s = OC.OA$

$= \text{area of the rect. } OABC$

Thus, the distance covered by the particle in the interval of time t, is respresented by the area of the rectangle OABC (Fig 8.2), where OB is the velocity-time graph.

Now, consider the case of a particle starting with an initial velocity u and moving along a straight line with a uniform acceleration f and acquiring the velocity v in an interval of time t, thereby describing a distance s. Surely, $v = u + ft$.

The tabulated values of the velocity V at the end of different intervals of the time T that successively elapsed is as follows :

Time : T	0	1	2	3	......	t
Velocity : V	u	$u+f$	$u+2f$	$u+3f$	......	$u+ft=v$

The graph of the velocity V against time T is the slanting straight line CB as shown in Fig. 8.3.

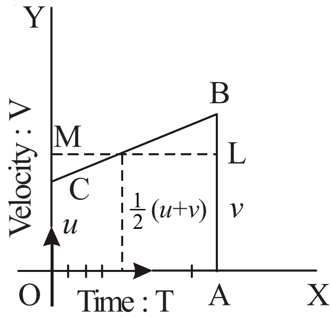


Fig. 8.3

The distance s described by the particle in the interval of time t is represented by the area of the trapezium of OABC (Fig 8.3), where CB is the velocity-time graph.

Thus, s = area of the trapezium OABC (= area of the rectangle OALM)

$$= \frac{1}{2}(\text{OC} + \text{AB})\text{OA}$$

$$= \frac{1}{2}(u+v)\cdot t \quad [\because \ \text{OA} = t,\ \text{OC} = u,\ \text{AB} = v]$$

$$= \frac{1}{2}(u+u+ft)\cdot t$$

i.e., $s = ut + \frac{1}{2}ft^2$.

III. To establish the formula $v^2 = u^2 + 2fs$.

We know that,

$$v = u + ft \quad \text{...} (1)$$

and $$s = ut + \frac{1}{2}ft^2 \quad \text{.......................................} (2)$$

Squaring both sides of (1), we get

$$v^2 = u^2 + 2uft + f^2t^2$$

$$= u^2 + 2f\left(ut + \frac{1}{2}ft^2\right)$$

$$= u^2 + 2fs \quad [\because \text{ of } (2)]$$

8.8 Distance covered in any particular second

Let u be the initial velocity of a particle moving along a straight line with uniform acceleration f.

Then, distance described by the particle in n seconds = $un + \frac{1}{2}fn^2$

and distance described by the particle in $(n-1)$ seconds $= u(n-1)+\frac{1}{2}f(n-1)^2$

Thus, if s_n denote the distance described by the particle in the n^{th} second of its motion, then

$$s_n = \left(un + \frac{1}{2}fn^2\right) - \left\{u(n-1)+\frac{1}{2}f(n-1)^2\right\}$$

$$\Rightarrow \; s_n = u + \frac{1}{2}f(2n-1)$$

Example 1 : Convert the velocity 60 miles/hr. in ft/sec.

Solution : 60 miles/hr. $= \frac{60 \times 1760 \times 3}{60 \times 60}$ ft/sec = 88 ft/sec.

Example 2 : Convert the velocity 80 m/sec in Km/hr.

Solution : Given velocity $= \frac{80 \times 60 \times 60}{1000}$ Km/hr = 288 Km/hr.

Example 3 : Convert the acceleration 432 Km/hr^2 in cm/sec^2.

Solution : Given acceleration = 432 Km/hr^2 $= \frac{432 \times 1000 \times 100}{60 \times 60 \times 60 \times 60}$ cm/sec^2 $= \frac{1}{3}$ cm/sec^2.

Example 4 : A body is moving with uniform velocity 60 ft/sec. How far it will travel in 20 seconds ?

Solution : Here $v = 60$ ft/sec, $t = 20$ secs, $s = ?$

From $s = vt$, we get

$s = 60 \times 20$

$= 1,200$ ft.

Example 5 : A body is moving with uniform velocity 20 m/sec. How long it will taken to travel 2 Km ?

Solution : Here $v = 20$ m/sec, $s = 2$ Km $= 2000$ m, $t = ?$

From $s = vt$, we get

$2000 = 20\,t \Rightarrow t = 100$ secs.

Example 6 : A train starts from rest with a acceleration 3 m/sec^2. What will be its velocity after 10 seconds and what is the distance travelled by the train during it ?

Solution : Here $u = 0$, $f = 3$ m/sec^2, $t = 10$ secs.

$v = ?$ and $s = ?$

From $v = u + ft$, we get

$v = 0 + 3 \times 10 = 30$ m/sec

Again from $s = ut + \frac{1}{2} ft^2$, we get

$$s = 0 \times 10 + \frac{1}{2} 3 \times 10^2$$

$= 150$ m.

Example 7 : A car is moving with a velocity 60 miles/hr. is brought to rest by using brake after traversing 121 ft. What is its retardation ?

Solution : Here $u = 60$ miles/hr = 88 ft/sec

$s = 121$ ft, $v = 0$, $f = ?$

From $v^2 = u^2 + 2fs$, we get

$0 = 88^2 + 2f \times 121$

$$\Rightarrow \quad f = -\frac{88 \times 88}{2 \times 121} = -32 \text{ ft/sec}^2 \text{ (– ve for retardation)}$$

Hence, retardation of the car is 32 ft/sec^2.

Example 8 : A particle moving in a straight line with uniform acceleration, describes a distance 81 cm and attains a velocity of 24 cm/sec in 3 seconds. Find the initial velocity and acceleration.

Solution : Here $v = 24$ cm/sec, $s = 82$ cm, $t = 3$ secs, $u = ?$ and $f = ?$

From the formulae $v = u + ft$ and $s = ut + \frac{1}{2}ft^2$,

we have

$24 = u + 3f$..(1)

and $81 = 3u + \frac{9}{2}f$...(2)

Solving (1) and (2), we get

$u = 30$ and $f = -2$

Hence the initial velocity is 30 cm/sec and the acceleration is – 2 cm/sec^2 (i.e. retardation is 2 cm/sec^2).

Example 9 : A particle moving with uniform acceleration in a straight line describes 25 cm in the third second and 55 cm in the sixth second of its motion. Find the initial velocity and acceleration.

Solution : Let u cm/sec be the initial velocity and f cm/sec^2 be the acceleration of the particle.

Then using the formula $s_n = u + \frac{1}{2} f(2n-1)$, we have

$$25 = u + \frac{1}{2} f(2 \times 3 - 1)$$

$$\Rightarrow \quad 25 = u + \frac{5}{2} f \text{ ....(1)}$$

and $\quad 55 = u + \frac{1}{2} f(2 \times 6 - 1)$

$$\Rightarrow \quad 55 = u + \frac{11}{2} f \text{ ....(2)}$$

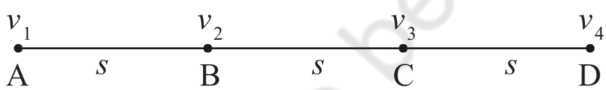
Solving (1) and (2), we get

$u = 0$ and $f = 10$

Hence, the initial velocity = 0 and the acceleration = 10 cm/sec^2.

Example 10 : If a particle moving under uniform acceleration describes successive equal distances in times t_1, t_2, t_3, prove that $\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}$

Solution :



Let the particle describe equal distances AB, BC, CD each equal to s, in times t_1, t_2, t_3 respectively. Let v_1, v_2, v_3, v_4 be the velocities of the particle at the points A, B, C, D respectively.

Considering the motion of the particle over the distances AB, BC, CD and using the formula $s = \frac{1}{2}(u + v)t$, we have

$$s = \frac{1}{2}(v_1 + v_2)t_1 \Rightarrow \frac{s}{t_1} = \frac{1}{2}(v_1 + v_2) \text{ ....(1)}$$

$$s = \frac{1}{2}(v_2 + v_3)t_2 \Rightarrow \frac{s}{t_2} = \frac{1}{2}(v_2 + v_3) \text{ ....(2)}$$

$$s = \frac{1}{2}(v_3 + v_4)t_3 \Rightarrow \frac{s}{t_3} = \frac{1}{2}(v_3 + v_4) \text{ ....(3)}$$

From (1), (2) and (3), we get

$$\frac{s}{t_1} - \frac{s}{t_2} + \frac{2}{t_3} = \frac{1}{2}(v_1 + v_2) - \frac{1}{2}(v_2 + v_3) + \frac{1}{2}(v_3 + v_4)$$

$$\Rightarrow \quad s\left(\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}\right)=\frac{1}{2}(v_1+v_4) \text{ .... (4)}$$

Again, considering the motion of the particle over the whole distance AD end using the same formula, we have

$$3s=\frac{1}{2}(v_1+v_4)(t_1+t_2+t_3)$$

$$\Rightarrow \quad \frac{3s}{t_1+t_2+t_3}=\frac{1}{2}(v_1+v_4) \text{ .... (5)}$$

From (4) and (5), we have

$$s\left(\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}\right)=\frac{3s}{t_1+t_2+t_3}$$

$$\Rightarrow \quad \frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{3}{t_1+t_2+t_3}.$$

Example 11 : If a body, falling freely from the top of a building reaches the ground in 10 seconds, find the height of the building and the velocity with which the body will strike the ground ($g = 9.8$ m/sec^2)

Solution : Here $u = 0$, $f\,(=g) = 9.8$ m/sec^2, $t = 10$ secs.

height, $h\,(=s) = ?$ and $v = ?$

From $s=ut+\frac{1}{2}ft^2$, we get

$$h=0+\frac{1}{2}\times 9.8\times 10^2=490 \text{ m.}$$

Again from $v = u + ft$, we get

$v = 0 + 9.8 \times 10 = 98$ m/sec.

Example 12 : A particle is projected vertically upwards with a given velocity u. How high and how long will it rise ? Find also the whole time of flight.

Solution : Here initial velocity $= u$, final velocity, $v = 0$, acceleration, $f = -g$

Let H be the greatest height, the particle can attain.

Then from $v^2 = u^2 + 2fs$, we get

$0 = u^2 + 2(-g)$ H

$$\Rightarrow \quad \text{H}=\frac{u^2}{2g}$$

Let t be the time of rising, i.e., the time to reach the greatest height. Then from $v = u + ft$, we get

$$0 = u - gt \quad \Rightarrow \quad t = \frac{u}{g}.$$

Let T be the whole time of flight, i.e., the time taken to return to the point of projection.

Here the distance travelled, $s = 0$.

Then from

$$s = ut + \frac{1}{2} ft^2, \text{ we get}$$

$$0 = u\mathrm{T} - \frac{1}{2} g\mathrm{T}^2$$

$$\Rightarrow \quad \mathrm{T} = 0 \quad \text{or} \quad T = \frac{2u}{g}$$

The first value T = 0 refers to the starting point and the second value $\mathrm{T} = \frac{2u}{g}$ is the time taken to reach the point of projection after reaching the highest point. (i.e., the whole time of flight)

Note : For falling bodies, $f = g$ and for bodies projected vertically upwards, $f = -g$.

Example 13 : From a balloon ascending with a velocity 32 ft/sec, a stone is let fall and reaches the ground in 10 secs. How high was the balloon when the stone was dropped ?

Solution : At the instant when the stone was dropped, it was moving with the velocity of the balloon, i.e. 32 ft/sec upwards

Here, $u = -32$ ft/sec, $f = 32$ ft/sec^2, $t = 10$ sec, $h = ?$

$\therefore$ from $h = ut + \frac{1}{2} gt^2$, we get

$$h = -32 \times 10 + \frac{1}{2} \times 32 \times 10^2 = 1280 \text{ ft.}$$

EXERCISE 8.1

1. Convert the following velocities in ft/sec.

   (i) 60 miles/hr (ii) 75 miles/hr (iii) 20 yards/min

2. Convert the following velocities in cm/sec.

   (i) 90 kilometres per hour (ii) 30 metres per minute (iii) 10 Km/min.

3. Convert the following velocities in cm/sec

   (i) 80 Km/hr (ii) 15 Km/min (iii) 30 cm/

4. Convert the following velocities in Km/hr.

   (i) 100 m/sec (ii) 300 m/min (iii) 50 m/

5. Convert the following acceleration in ft/sec^2.

   (i) 1 mile/ ^{2} (ii) 5400 miles/hr^2 (iii) 300 yds/min^2

6. Convert the following in m/sec^2

   (i) 980 cm/ ^{2} (ii) 1296 Km/hr^2 (iii) 360 Km/min^2

7. A car is travelling with uniform velocity 50 m/sec. How far it will travel in 10 secs and how long will it take to travel 2 Kms ?
8. If a train travels 108 Kms in 3 hrs, what is the average velocity in m/sec ?
9. If the velocity of a train 100 m long is 1 Km/min, how many seconds will it take to cross a telegraph post ?
10. A bus starts moving from rest with an acceleration 2 m/sec^2. What will be its velocity after 15 seconds and how far it will travel during the time and during the next 5 seconds ?
11. The velocity of a car increases from 20 m/sec to 30 m/sec while it travels a distance of 25 m. What is its acceleration ?
12. An arrow can rise 160 m vertically upwards ; what is its velocity at the time of shooting ? ($g = 9.8$ m/sec^2)
13. If a body falling freely from the top of a building, reaches the ground in 5 seconds. Find the height of the building and the velocity with which it strikes the ground. ($g = 32$ ft/sec^2)
14. A body, starting from rest and moving with uniform acceleration, describes 27 m in 3 secs ; what is its velocity at the end of 10 seconds ? Find also the distance traversed in the 5th second of its motion.
15. A stone is dropped into a well and reaches the surface of the water is 3 seconds ; with what velocity will it reach the surface ? Find also the depth of the well. ($g = 32$ ft/sec^2)
16. From a balloon ascending with a velocity of 10 ft/sec, a stone is let fall and reaches the ground in 15 seconds. How high was the balloon when the stone was dropped ?
17. A falling particle in the last second of its fall, passes through 112.7 m. How long did it fall ? Find also the height from which it fell. ($g = 9.8$ m/sec^2)

18. A stone is dropped into a well and the sound of the splash is heard in $7\frac{7}{10}$ seconds. If the velocity of sound be 1120 ft/sec, find the depth of the well. ($g = 32$ ft/sec^2)
19. A stone is thrown vertically upwards with a velocity of 42 m/sec. Find
    (i) its velocity at the end of 3 seconds.
    (ii) the maximum height attained and the time taken.
20. A point starts from rest and moves with a uniform acceleration of 2 m/sec^2 ; find the time taken by it to traverse the first, second and third metre respectively.
21. A particle is moving with uniform retardation ; in the third and eight second after starting it moves through 255 and 225 cm respectively ; find its initial velocity and its retardation.
22. A body starting from rest and moving with uniform acceleration describes 999 cm in the fifth second ; find its acceleration.
23. A point starts with a velocity of 100 cm/sec and moves with a retardation of 10 cm/sec^2. When will its velocity be zero and how far will it have gone ?
24. A body starts with a velocity of 5 cm/sec and moves with a uniform acceleration of 5 cm/sec^2. How far it will move after 4 seconds ?
25. A cyclist driving with uniform velocity of 20 ft/sec is 84 ft behind an engine which is just starting from rest with uniform acceleration 2 ft/sec^2. When will the cyclist meet the engine ? Explain the double answer.
26. A car moving in a straight path, with uniform acceleration passes over 7 metres in the 2nd second and 11 metres in the 4th second of its motion. Find the initial velocity and acceleration.
27. A particle starts with an initial velocity u and passes successively over the two-halves of a given distance with accelerations f_1 and f_2 respectively. Show that the final velocity is the same as if the whole distance were traversed with uniform acceleration $\frac{1}{2}(f_1 + f_2)$.
28. If s is the space described in t seconds and s' during the next t' seconds by a particle moving in a straight line with uniform acceleration f, show that

$$f = \frac{2\left(\frac{s'}{t'} - \frac{s}{t}\right)}{t + t'}.$$

29. A particle moving with constant acceleration from A to B in a straight line AB has velocities u and v at A and B respectively. Prove that its velocity at C, the mid-point of AB, is $\sqrt{\frac{u^2 + v^2}{2}}$.

ANSWERS

1. (i) 88 ft/sec. (ii) 110 ft/sec (iii) 1 ft/sec.

2. (i) 2500 cm/sec. (ii) 50 cm/sec. (iii) $16666\frac{2}{3}$ cm/sec.

3. (i) $22\frac{2}{9}$ m/sec (ii) 250 m/sec ($\frac{3}{10}$ m/sec.

4. (i) 360 Km/hr. (ii) 18 Km/hr. (iii) 180 Km/hr.

5. (i) $1\frac{7}{15}$ ft/sec^2 (ii) $2\frac{1}{5}$ ft/sec^2 (iii) $\frac{1}{4}$ ft/sec^2

6. (i) 9.8 m/sec^2 (ii) 0.1 m/sec^2 (iii) 100 m/sec^2

7. 500 m and 40 secs.

8. 10 m/sec

9. 6 secs.

10. 30 m/sec, 225 m, 175 m

11. 10 m/sec^2

12. 56 m/sec

13. 400 ft, 160 ft/sec

14. 60 m/sec, 27 m

15. 90 ft/sec^2, 144 ft.

16. 3450 ft.

17. 12 secs ; 705.6 m.

18. 784 ft.

19. (i) 12.6 m/sec (ii) 90 m ; $4\frac{2}{7}$ secs.

20. 1, $\sqrt{2}-1$, $\sqrt{3}-\sqrt{2}$ secs.

21. 270 cm/sec ; 6 m/sec^2

22. 222 cm/sec^2

23. 10 secs ; 500 cm

24. 60 cm.

25. 6 secs and 14 secs

26. 4 m/sec and 2 m/sec^2

8.9 Motion of a body having more than one velocity at a time

You have discussed the motion of a body when it possesses only one velocity which may be uniform or non-uniform. Now, one question before us is the case of a body having more than one velocity at a time. In our common life also, we come across such situations.

Imagine an empty plastic bottle floating on the water, flowing steadily from north to south at a uniform rate say, u Km/hr. The bottle will move at the rate of the flow of the water in the direction of the current. Suppose the wind is blowing uniformly from west to east at the rate of say, v Km/hr.

Under these conditions, relative to the bed of the river at what rate is the bottle moving and as observed from a fixed point on the western bank of the river, in which direction will the bottle move when it is having the two simultaneous velocities viz.

(i) The velocity due to the current at the rate of u Km/hr from north to south.

(ii) Velocity due to the wind at the rate of v Km/hr from west to east ?

We shall not give answers to these questions now. We are simply probing the situations. Had there been no wind, the veolcity of the bottle would have been u Km/hr from north to south. And had there been no current, the veloctity of the bottle would have been v Km/hr from west to east.

Similarly, a body may have three velocities simultaneously. For example, imagine the case of a sailor climbing the vertical mast of a ship at the rate of say, u m/sec at a time when the ship is sailing due south at the rate of say, v m/sec and the wind is blowing due east at the rate of say, w m/sec.

Now, relative to the sea bed, how fast is the sailor moving and in what direction ?

These are the questions that we shall try to answer in the following sections. We shall also examine the situation where a body has a number of simultaneous velocities more than three in number.

8.10 The Parallelogram Law of Velocities

Statement :

If a body possesses two simultaneous velocities which can be represented in magnitude and direction by two adjacent sides of a parallelogram drawn from one of its angular points, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from the angular point.

In the adjoining diagram Fig. 8.4, O is the position of a particle which has two simultaneous velocities u and v along the direction indicated by the arrow heads.

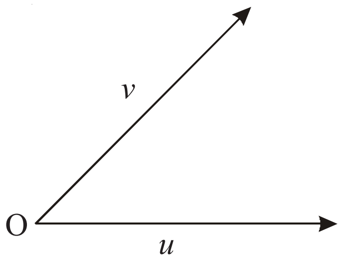


Fig. 8.4

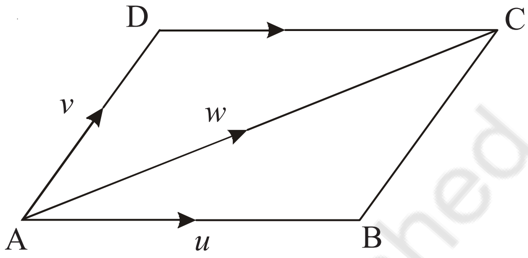


Fig. 8.5

Again in Fig. 8.5, A is the angular point from which the two adjecent sides AB and AD of the parallelogram ABCD are drawn such that they respectively represent the velocities u and v of the particle at O both in magnitude and direction.

Under this condition the parallelogram law of velocities states that the diagonal AC drawn from the point A represents the resultant of u and v both in magnitude and direction. If w is the resultant velocity, symbolically we write

$$\vec{u} + \vec{v} = \vec{w} \quad \text{in the form of vector sum.}$$

We also use the notation $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$

Or, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (since $\overrightarrow{BC} = \overrightarrow{AD}$)

The following is not a theoretical proof of the law. It is only an elucidation of the same.

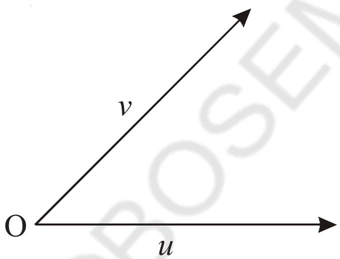


Fig. 8.6

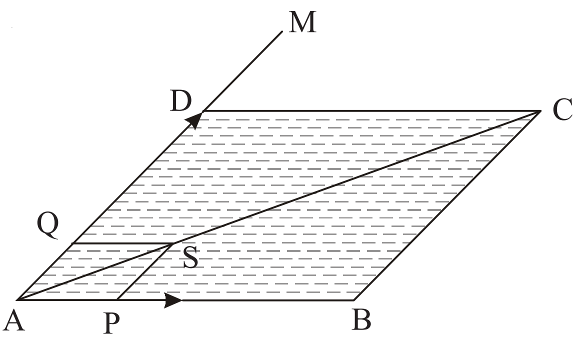


Fig. 8.7

Imagine a perfectly even and cylindrical crayon AB of length u units. As shown in Fig. 8.7. This crayon is placed parallel to the velocity u which is one of the simultaneous velocities of the particle at O as shown in Fig. 8.6. AM is a straight line parallel to the other velocity v of the particle at O.

Keeping the end A always in contact with the line AM, let the crayon slide parallel to itself at the uniform rate of v units per unit time so that at the end of the unit of time it occupies the position DC and by doing so it sweeps out a region (the boundary being a parallelogram) shown by the shaded portion.

Now imagine a particle placed at the end of A of the crayon and moving along its length at the rate of u units per unit time while the crayon itself is sliding from the position AB to the position DC in that unit of time.

Now due to the combined effect of the two simultaneous velocities, at the end of the unit of the time the particle reaches the other end B of the crayon while B itself reaches C at the same instant.

The particle which was initially at A is now at C at the end of unit of time.

Thus AC is the actual distance covered by the particle in that unit of time and also the actual direction which the particle moves is along AC.

If w denotes the resultant of u and v then, we say that AC represents w in magnitude and direction.

Using vector symbols we write

$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

$$\text{Or,} \quad \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad (\text{since } \overrightarrow{AD} = \overrightarrow{BC})$$

$$\text{Or,} \quad \vec{u} + \vec{v} = \vec{w}$$

We can see that at an intermediate time say, at the end of $\frac{1}{n}^{\text{th}}$ of the unit of time, the particle will be at some point P on AB while the end A will be at some point Q on AD such that, $\frac{AP}{AB} = \frac{1}{n}$.

Completing the parallelogram APSQ and using properties of similar triangles we can show that,

$$\frac{AP}{AB} = \frac{AQ}{AD} = \frac{AS}{AC} = \frac{1}{n}$$

In fact at the end of the $\frac{1}{n}^{\text{th}}$ of the unit of time the particle is at S.

In short, the position of the particle at any instant of time is on the diagonal AC.

Special Cases :

When u and v along the same line, two cases aries,

(i) When they are along the same direction :

In this case the magnitude of the resultant is their sum and it is along the common direction.

i.e. $w = u + v$

(ii) When they are in opposite directio

In this case the magnitude of the resultant is their difference i.e. $|u - v|$ and it is in the direction of the numerically greater velocity.

i.e. $w = u - v$, when $u > v$

$w = v - u$, when $v > u$

Giving, $|w| = |u - v|$.

8.11 Composition and Resolution of Velocities

You have seen that the resultant of two simultaneous velocities is a velocity called the resultant of the two velocities. On the other hand the two velocities are called the components of the resultant velocity. Further, it has been seen that the law governing the composition of the two velocities is given by the law of the parallelgram of the velocities.

Now, one pertinent question before us is, "Since composition of two velocities is a mathematical operation, like other operations does it have a reverse operation ?" Given any velocity u, is it possible to find two component velocities along two arbitrary directions such that the resultant of these two is the original velocity u ?

The answer is, yes, it is possible. Infact a given velocity can be split endlessly along two directions in infinitely many ways. But here we shall consider the special case of splitting a velocity along two mutually perpendicular directions.

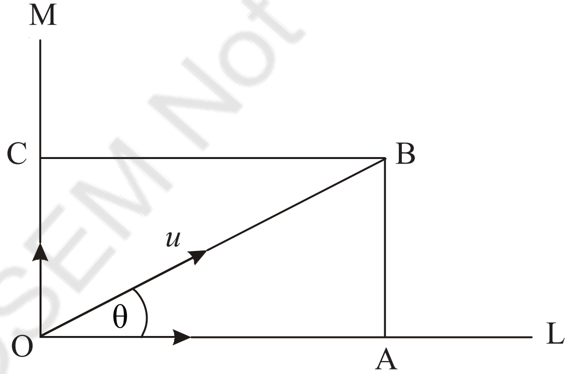


Fig. 8.8

As shown in Fig. 8.8, let OB represent a velocity u and let OL be a line making an angle θ with OB. Draw OM perpendicular to OL. With OB as diagonal we construct the rectangle OABC. Since OABC is a rectangle and hence a parallelogram, the resultant of the velocities represented by OA and OC is represented by OB. i.e. the resultant is u.

This shows that the components of u along OL and OM are represented by OA and OC respectively.

Thus, component of u along $OL = OA = OB \times \frac{OA}{OB}$

$$= OB \cos\theta$$
$$= u \cos\theta$$

And component of u along $OM = OC = OB \times \frac{OC}{OB}$

$$= OB \sin\theta$$
$$= u \sin\theta$$

These mutually perpendicular components of the velocity u are called the resolved parts or the resolutes along these directions. In general, the process of splitting a given velocity is known as resolution of the velocity which is the reverse process of composition of velocities.

Deduction :

The component of a velocity u in a direction making an angle θ with its direction is $u \cos\theta$. Since $\cos\theta$ is maximum when $\theta = 0^o$ and minimum when $\theta = 90^o$, we see that the maximum component of a velocity u is $u \cos 0^o$ i.e. u itself and the minimum component is $u \cos 90^o$ i.e. O. Thus a velocity has no component in a direction perpendicular to itself. This fact is very important in view of the following consideration.

Imagine a velocity u represented by OA in magnitude and direction.

Let OL and OM be two arbitrary directions making angles α and β respectively with OA as shown in Fig. 8.9. AB and AC are perpendiculars from A on OL and OM respectively.

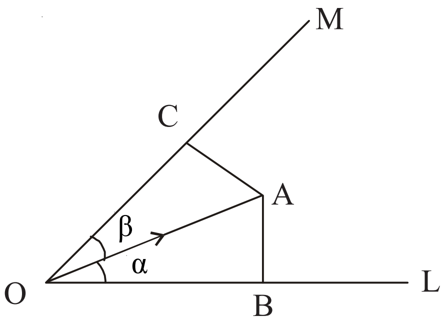


Fig. 8.9

Now component of u along OL $= u \cos\alpha$ and it is represented by OB.

And component of u along OM $= u \cos\beta$ and it is represented by OC.

But the resultant of these two components represented by OB and OC is not u, showing that u represented by OA cannot be replaced by $u \cos\alpha$ along OL and $u \cos\beta$ along OM.

In fact, you will see that the actual components of u along OL and OM which can completely replace u, are respectively $\frac{u\sin\beta}{\sin(\alpha+\beta)}$ and $\frac{u\sin\alpha}{\sin(\alpha+\beta)}$.

[This result can be easily derived using the sine formula of trigonometry and the triangle law of velocities.]

However, if $\alpha+\beta = 90^\circ$ then,

Component of u along $\text{OL} = \frac{u\sin\beta}{\sin 90^\circ}$

$$= u\sin(90^\circ - \alpha)$$

$$= u\cos\alpha$$

And the component of u along $\text{OM} = \frac{u\sin\alpha}{\sin 90^\circ}$

$$= u\sin(90^\circ - \beta)$$

$$= u\cos\beta$$

8.12 Resultant of two Velocities

The resultant of two velocities u and v of a particle, in directions which are inclined to one another at an angle α may be easily obtained.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the velocities u and v so that $\angle AOB = \alpha$ (Fig. 8.10). Complete the parallelogram OACB. Then, the diagonal $\overrightarrow{OC}$ will represent the resultant veloctity V of u and v.

Let OC make an angle θ with OA.

Draw CD perpendicular to $\overleftrightarrow{OA}$.

Now, in the right ΔACD, $\angle CAD = \alpha$ (why ?) so that

$$AD = AC \times \frac{AD}{AC} = AC\cos\alpha$$

and $$CD = AC \times \frac{CD}{AC} = AC\sin\alpha.$$

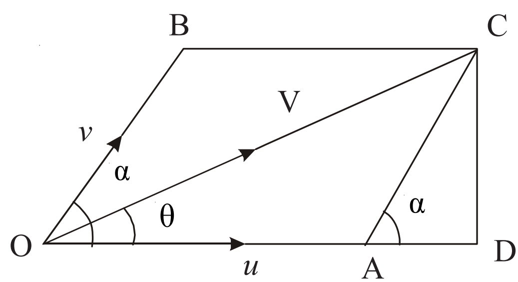


Fig. 8.10

By Pythagoras Theorem,

$$OC^2 = OD^2 + CD^2$$

$$= (OA + AC\cos\alpha)^2 + (AC\sin\alpha)^2$$

$$= OA^2 + 2 \times OA \times AC \cos\alpha + AC^2(\cos^2\alpha + \sin^2\alpha)$$
$$= OA^2 + AC^2 + 2 \times OA \times AC \cos\alpha$$
$$= OA^2 + OB^2 + 2 \times OA \times OB \cos\alpha \qquad [\because AC = OB]$$

Thus, the magnitude of the resultant velocity is given by

$$\boxed{V^2 = u^2 + v^2 + 2uv\cos\alpha}$$

Furhter, $\tan\theta = \dfrac{CD}{OD} = \dfrac{AC\sin\alpha}{OA + AC\cos\alpha}$

$$= \frac{OB\sin\alpha}{OA + OB\cos\alpha}$$

$$= \frac{v\sin\alpha}{u + v\cos\alpha}$$

This angle θ determines the direction of the resultant velocity.

Corollary 1 : If $\alpha = 90°$ (i.e. u and v are perpendicular to one another), then

$$V^2 = u^2 + v^2 \text{ and } \tan\theta = \frac{v}{u} \quad [\because \cos 90° = 0 \text{ and } \sin 90° = 1]$$

Corollary 2 : If $\alpha = 0°$ (i.e. u and v have the same direction), then

$$V^2 = u^2 + v^2 + 2uv \quad [\because \cos 0° = 1]$$
$$= (u + v)^2$$
$$\therefore \quad V = u + v$$
$$\text{and} \quad \theta = 0°$$

Corollary 3 : If $\alpha = 180°$ (i.e. u and v have the opposite directions), then

$$V^2 = u^2 + v^2 - 2uv \quad [\because \cos 180° = -1]$$
$$= (u - v)^2$$
$$\therefore \quad V = \begin{cases} u - v & \text{when } u > v \\ v - u & \text{when } v > u \end{cases}$$

8.13 Triangle Law of Velocities

From the parallelogram law of velocities we have seen that if a body has two simultaneous velocities u and v which can be respectively represented in magnitude and direction by the sides AB and BC of a triangle ABC, then their resultant is represented by the third side AC in magnitude and direction.

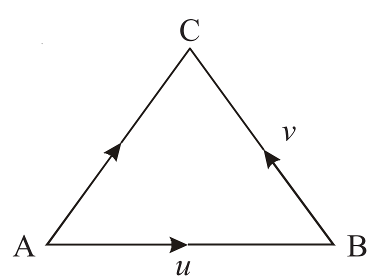


Fig. 8.11

From this, a law called the triangle law of velocities, can be deduced. This runs as follows :

Statement :

If a body has three simultaneous velocities which can be represented in magnitude and direction by the three sides of a triangle taken in order, then the resultant velocity is zero thereby keeping the body at rest.

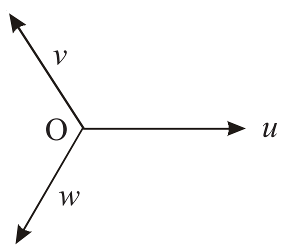


Fig. 8.12

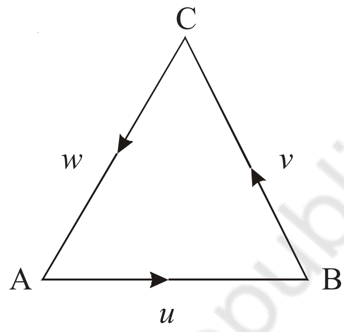


Fig. 8.13

In Fig. 8.12, O is the point where the particle having the three simultaneous velocities *u, v* and *w* is situated. In Fig 8.13, the sides AB, BC, CA of the triangle ABC in this order respectively represent *u, v* and *w* in magnitude and direction. Now, the resultant of the velocities *u* and *v* represented by AB and BC is represented by AC in magnitude and direction. Hence, the final resultant is represented by the resultant of the velocities represented by AC and CA. Surely, they are equal in magnitude and opposite in direction.

Thus, the final resultant velocity is zero and hence the body is at rest.

We express the result as $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

This result enables us to extend the principle to what is known, in short, by the polygon of velocities.

8.14. Polygon of Velocities.

Statement :

Let ABCDEF be a polygon and let there be a particle having simultaneous velocities which can be represented in magnitude and direction by the sides AB, BC, CD, DE and EF. Then the resultant velocity is represented in magnitude and direction by the side AF.

Now, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= \overrightarrow{AD} + \overrightarrow{DE} + \overrightarrow{EF}$

$= \overrightarrow{AE} + \overrightarrow{EF}$

$= \overrightarrow{AF}$

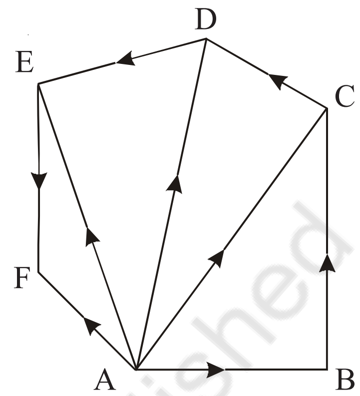


Fig. 8.14

Had there been another velocity represented by FA, then the resultant would have been 0 and the body would have been at rest.

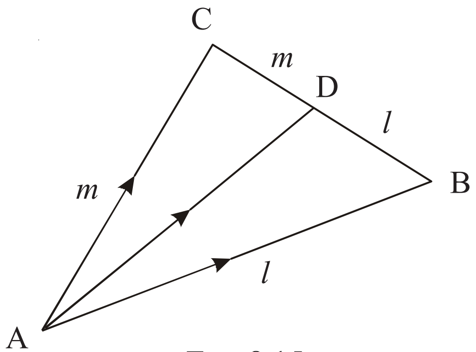


Fig. 8.15

In a triangle ABC if D is a point on the side BC such that it divides BC internally in the ratio $l : m$ then, resultant of the two velocities represented in magnitude and direction by lAB and mAC is represented by $(l + m)$ AD in magnitude and direction.

Now, $\frac{BD}{DC} = \frac{l}{m} \Rightarrow l DC = m BD$ ....(i)

From the triangle of velocities we get,

$$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$$

$$\Rightarrow l\overrightarrow{AB} + l\overrightarrow{BD} = l\overrightarrow{AD} \quad \text{.. (ii)}$$

And, $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$$\Rightarrow m\overrightarrow{AC} + m\overrightarrow{CD} = m\overrightarrow{AD} \quad \text{.. (iii)}$$

Adding (ii) and (iii)

$$(l\overrightarrow{AB} + m\overrightarrow{AC}) + (l\overrightarrow{BD} + m\overrightarrow{CD}) = (l + m)\overrightarrow{AD}$$

$\Rightarrow l\overrightarrow{AB} + m\overrightarrow{AC} + (l\overrightarrow{BD} - m\overrightarrow{DC}) = (l + m)\overrightarrow{AD}$ [since $\overrightarrow{CD}$ & $\overrightarrow{DC}$ are opposite in direction]

$\Rightarrow l\overrightarrow{AB} + m\overrightarrow{AC} = (l + m)\overrightarrow{AD}$ [since $l\overrightarrow{BD}$ & $\overrightarrow{DC}$ are equal in magnitude by (i) but opposite in direction]

8.15. Determination of the resultant of a number of simultaneous co-planar velocities of particle.

If a body possesses a number of co-planar velocities simultaneously, their resultant can be determined on replacing every one of them by its resolved parts along two conveniently chosen or mutually perpendicular lines and compounding them together.

In Fig. 8.16, LOL′ and MOM′ are two mutually perpendicular straight lines.

The particle at O has velocities u_1, u_2, u_3.....................u_n making respectively angles θ_1, θ_2, θ_3 ....θ_n with OL.

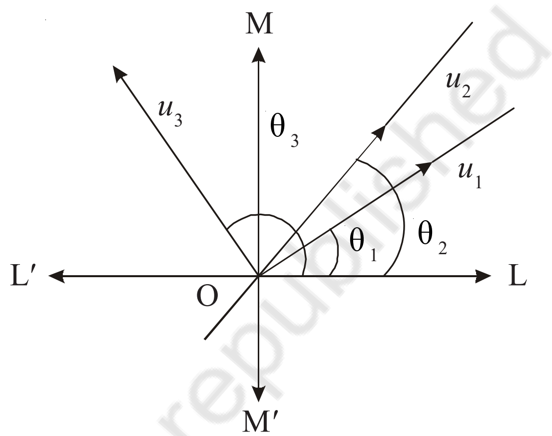


Fig. 8.16

Now u_1 can be replaced by $u_1\cos\theta_1$ along OL and $u_1\sin\theta_1$ perpendicular to it. Similarly, u_2 can be replaced by $u_2\cos\theta_2$ along OL and $u_2\sin\theta_2$ perpendicular to it. Similarly, all the other velocities can be replaced by their components.

If X and Y respectively denote the algebraic sums of all components along OL and perpendicular to it, i.e. along OM then,

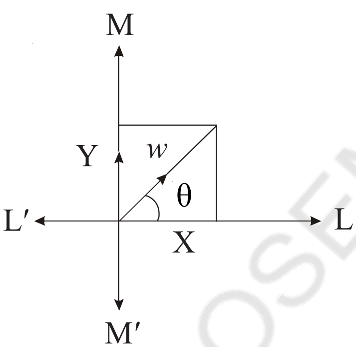


Fig. 8.17

$$X = u_1\cos\theta_1 + u_2\cos\theta_2 + ... + u_n\cos\theta_n = \sum_{i=1}^{n} u_i\cos\theta_i$$

$$\text{And, } Y = u_1\sin\theta_1 + u_2\sin\theta_2 + ... + u_n\sin\theta_n = \sum_{i=1}^{n} u_1\sin\theta_i$$

Giving the resultant velocity say, w on using the relation

$$w^2 = X^2 + Y^2 \text{ .... (i)}$$

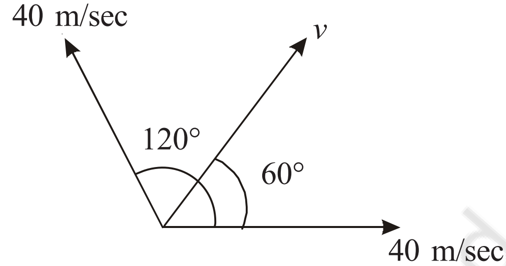
and the angle made by w with the line OL from

$$\tan\theta = \frac{Y}{X} \text{ .... (ii)}$$

Example 1 : Find the resultant of two equal velocities of 40m/sec inclined at an angle of 120º to each other.

Solution :

Let v be the resultant velocity, then,



$$v^2 = 40^2 + 40^2 + 2\times 40\times 40\times \cos 120^o$$
$$= 40^2 + 40^2 + 2\times 40\times 40\times\left(-\frac{1}{2}\right)$$
$$= 40^2 + 40^2 - 40^2$$
$$\Rightarrow v = 40\,\text{m/sec.}$$

If v makes an angle θ with one of the component velocities, then

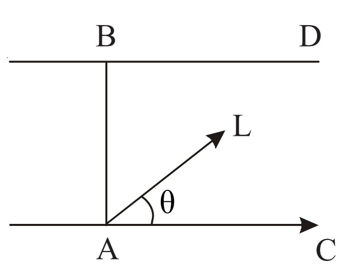
$$\tan\theta = \frac{40\sin 120^o}{40 + 40\cos 120^o}$$
$$= \frac{40\times\frac{\sqrt{3}}{2}}{40 - 40\times\frac{1}{2}} = \frac{2\sqrt{3}}{20} = \sqrt{3}$$

$$\Rightarrow \tan\theta = \sqrt{3}$$
i.e. $\theta = 60^o$

Thus, the resultant is 40 m/sec in a direction equally inclined to both the component velocities.

Example 2 : A man who can swim in still water at the rate of 10 m/sec wishes to cross a river 100 m wide where water is flowing uniformly at the rate of 15 m/sec. Find the direction in which he should swim in order to cross the river in the shortest time. Also indicate the position on the opposite bank where he would land after remaining in water for the shortest time. Examine if it would be possible for him to cross the river by the shortest distance. Giving reason, state the possibility of such an option.

Solution : In the adjoining diagram, A is the position of the man on one bank of the river from where he is to swim and B is the point on the other bank just opposite to A so that AB = 100 m.



$\overrightarrow{AC}$ is the direction of the flow of water.

Let the man attempt to swim in the direction $\overrightarrow{AL}$ making an angle θ with AC.

Now, replacing the man's velocity along AL by its resolved parts along and perpendicular to AC we get, 10 cos θ m/sec along AC and 10 sin θ m/sec along AB.

Remembering that only the component 10 sin θ is responsible to carry the man across the river, as any velocity in the direction of the current has no effect in the perpendictular direction, let t seconds be the time taken to cross the river.

Then, $100 = 10 \sin\theta \times t$

$$\Rightarrow \quad t = \frac{10}{\sin\theta} \quad \text{...} \text{(i)}$$

Now, from (i) we see that t is minimum when sin θ is maximum. But the maximum value of sin θ is 1 and it happens when θ = 90°.

Thus, to cross the river by the shortest time the man has to swim perpendicular to the current and the time taken in swimming so is 10 seconds [taking sin θ = 1 in (i)]

However, he will not reach the other bank at the point B.

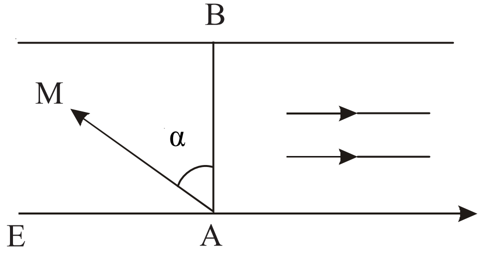
During this interval of 10 seconds, the man will be carried down the current with the velocity (15 + 10 cos 90°) m/sec. i.e. 15 m/sec.

If d be the distance carried down the current, then

$$d = 15 \times 10$$
$$= 150 \text{ m}$$

Thus, the man will land on the opposite bank at a point D where BD = 150 m.

In order to cross the river directly, the man should not be carried down the line AB (directly across the river)



In this case let the man swim in an upstream direction $\overrightarrow{AM}$ where ∠BAM = α as in the above diagram.

Resolving the velocity of the man along AB and AF, E being a point on the bank upstream, we get

10 cos α along AB

And, 10 sin α along AE.

Inorder that the man is not carried down the line AB, the upstream component of his velocity must neutralize the downward velocity of the current.

i.e. $10 \sin\alpha = 15$

$$\Rightarrow \quad \sin\alpha = \frac{3}{2} \qquad \text{which is impossible as } |\sin\theta| \le 1$$

Thus with his capacity of swimming at the rate of 10 m/sec he cannot cross the river by the shortest distance, i.e. he cannot cross the river directly.

In order to have this option of directly crossing the river, he must increase his velocity to more than 15 m/sec.

However, if his velocity is exactly 15 m/sec and if he swims exactly upstream he will neither be carried down nor cross the river. He will remain at A.

Example 3 : If the resultant of two simultaneous velocities of 15 m/sec and 30 m/sec is perpendicular to the smaller velocity, find the angle between them and also their resultant.

Solution : Let α be the angle between the two velocities.

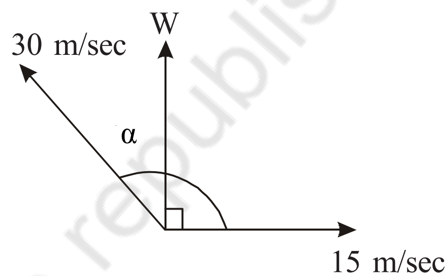
Since the resultant say, W, is perpendicular to the velocity of 15 m/sec we get

$$\tan 90^\circ = \frac{30\sin\alpha}{15+30\cos\alpha}$$

$$\Rightarrow \infty = \frac{30\sin\alpha}{15+30\cos\alpha}$$

$$\Rightarrow 15 + 30\cos\alpha = 0$$

$$\Rightarrow \cos\alpha = -\frac{1}{2} \quad \text{giving } \alpha = 120^\circ$$



Also, $W^2 = 15^2 + 30^2 + 2\times15\times30\cos120^\circ$

$$= 15^2 + 30^2 - 2\times15\times30\times\frac{1}{2}$$

$$= 15^2(1+2^2-2)$$

$\Rightarrow W = 15\sqrt{3}$ m/sec.

EXERCISE 8.2

1. Find the magnitude of the resultant of each of the following pairs of simultaneous velocities with the angle between them given therewith :
   (i) 30 cm/sec and 40 cm/sec, 90°
   (ii) 60 m/sec and 80 m/sec, 60°
   (iii) 5 m/sec and 10 m/sec, 45°
   (iv) 6 m/sec and 10 m/sec, 120°
   (v) 10 m/sec and 30 m/sec, 135°

2. Find the resolved parts of each of the following velocities whose inclination to one of the resolved parts is also given alongside:
   (i) 40 m/sec, 30° (ii) 70 m/sec, 60°

(iii) 60 m/sec, 45º (iv) 10 m/sec, 120º

(v) 20 m/sec, 135º

Interprete the cases in (iv) and (v).

3. A body has two equal (in magnitude) simultaneous velocities with 60° as the angle between them. Determine the resultant velocity and show that it is equally inclined to the velocities.

4. A man who can swim in still water at the rate of 8m/sec, wishes to cross a river 40 meters wide and whose current is flowing at the rate of 4 m/sec. Find the difference between the times taken by the man in crossing the river by the shortest time and shortest distance.

5. A body has two simultaneous velocities of 6 and 8 units, along two different directions. When a third velocity of 10 units is applied to the body it comes to rest. Find the angle between the first two velocities.

6. Find the maximum and the minimum resultants of the three simultaneous velocities of 5, 12 and 13 units possessed by a body.

7. Can the three simultaneous velocities of 2, 3 and 7 units keep a body at rest ? Give reason for your answer.

8. Find the magnitude and direction of the resultant of the following four simultaneous velocities of a body : 30 m/sec due East, 40 m/sec due North-East, 50 m/sec due North and 10 m/sec due North-West.

9. Find the resultant of three simultaneous velocities all equal in magnitude when the angle between each two of them is 120°.

10. A body has two simultaneous velocities which can be represented in magnitude and direction by 3AB and 2BC of a triangle ABC. Find the position of the point D on BC such that kAD represents the resultant of the two velocities, where k is a constant. State the value of k.

11. Giving reason state if a body having three simultaneous velocities 2u, 3u and 6u can have their resultant velocity zero.

ANSWER

1. (i) 50 cm/sec (ii) $20\sqrt{37}$ m/sec

(iii) $5\sqrt{5+2\sqrt{2}}$ m/sec (iv) $2\sqrt{19}$ m/sec

(v) $\sqrt{10\left(19-9\sqrt{2}\right)}$ m/sec

2. (i) $20\sqrt{3}$ m/sec, 20m/sec (ii) 35 cm/sec, $35\sqrt{3}$ cm/sec

(iii) $30\sqrt{2}$ cm/sec, each.

(iv) -5 m/sec, $5\sqrt{3}$ m/sec. The (–) ve sign before the first component shows that it is in a direction opposite to the given direction so that it makes 60º with the given velocity.

(v) $-5\sqrt{2}$ m/sec, $5\sqrt{2}$ m/sec.interpretation is similar to that of (iv)

3. $\sqrt{3u}$. The inclination of the resultant to each velocity is 30º

4. 0.8 sec (nearly)

5. 90°

6. Max. value = 30 units, when all the velocities are along the same line and same direction.

   Min. value = 0, when the velocities of 5 and 12 units are perpendicular to each other and the third is just opposite to the resultant of the first two velocities.

7. No. The maximum resultant of 2 and 3 units which is 5 units cannot balance the third velocity of 7 units.

8. $10\sqrt{17}\left(1+\sqrt{2}\right)$m/sec, making an angle θ north of east such that tan $\theta = \frac{5}{3}$.

9. 0.

10. The resultant is represented by 3AD where D divides BC in the ratio 2:1.

11. No Taking the pair of any two velocities of the three we see that the third velocity is always outside the limits of the maximum and minimum resultants of the first two velocities.

8.16 Kinetics

As stated earlier, Kinetics is the branch of Dynamics which deals with the physical aspects of the motion of a body. Naturally, in this case the mass of the body in motion has an important role to play.

It is really not out of the way if someone asks the question, "Similar to the discussion of motion with changing velocity, should the case of changing mass be considered?" Yes, this is surely within the scope of Kinetics. But, in our present study we shall not deal with case of changing mass. In other words, we shall assume that the body in motion is of constant mass.

In the following sections, we shall discuss the laws governing the motion of bodies and other related entities.

8.17 Motion and Momentum

When a body is in motion it acquires a certain property by virtue of its motion. Imagine a moving stone, when it is at rest it cannot harm anybody. But when it is in motion it can cause damage to materials or living beings. Due to the motion, there is no change in the mass of the body. Therefore, the causative factor of the acquisition of the property is the motion. Further, different masses with the same velocity have different amounts of the property.

This property which is depending both on the amount of the mass and velocity of a body is called the momentum of the body and it is measured by the product of the mass and the velocity of the body. It is a physical example of the product of a scalar and a vector. Surely, momentum is a vector quantity. The FPS unit of momentum is the pound-foot-per second which is sometimes known as the poundem. The CGS and the MKS units of momentum are the gm-cm per second and kg-metre per second respectively.

8.18 Types of Motion

When a body is in motion it is in one of the following four types of movement. They are,

i. Uniform motion along a straight line.

ii. Non-uniform motion along a straight line,

iii. Uniform motion along curves,

iv. Non-uniform motion along curves.

In this book, we consider only the first two types of motion which are termed as rectilinear motions.

Newton's laws of motion

For any body in the Universe, Sir Isaac Newton assigned two fundamental states of existence. One is the state of rest and the other is the state of motion of the type (i) stated above i.e. the state of uniform motion along a straight line. Other states of existence associated with the remaining three types of motion are taken as derivatives from these two fundamental states under the action of the external agents. His three laws of motion which form the foundation of mechanics are taken more or less like the axioms in Euclidean Geometry.

The laws with brief annotations are given below :

First law :

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by external forces to change that state.

This law is a qualitative law. In it implicitly the definition of a force is given. Further, for any material body it unfolds the body's innate tendency of preserving its original state which may be either state of rest or state of uniform motion along a straight line.

This tendency of unwillingness of automatic change of the original state is called the state of inertia. Thus, we have two types of inertia namely, the inertia of rest and the inertia of uniform motion along a straight line and a force is an external agent that changes or tends to change the state of rest or state of uniform motion along a straight line of a body.

In view of the affinity of this law to one of the forms of inertia, the 'law of inertia' is also used to denote this law.

Many common phenomena can be explained on the basis of this law. For example, when a stationary bus starts running, a sitting passenger is likely to fall backward while a passenger in a running bus is likely to fall forward when the bus suddenly stops. The reason for the first experience is that the lower part of the passenger's body has moved together with the bus while the upper part, because of inertia of rest is still stationary. The reason for the second case is the reverse of the first and here the inertia of motion of the upper part of the body of the passenger is coming into play.

Second law :

Imagine a body in the state of rest or in the state of uniform motion along a straight line. By the first law each body will maintain its momentum. If the momentum changes, then, it must have been caused by a force. For such a change of momentum, the second law states that *the time rate of change of momentum is proportional to the motive force and takes place in the direction of the straight line in which the force acts.*

When the mass of the body remains unchanged the change in momentum is caused only by the change in the velocity of the body and accordingly the second law can be reduced to the following form :

(The product of the mass and the time rate of change of velocity is proportional to the motive force.)

Using symbols, let m be the mass of the body whose velocity has undergone a change from u to v and consequently the momentum from mu to mv in an interval of time t during which it is acted upon continuously by the force P.

Then, the change in momentum in the interval of time t is $mv\text{-}mu$. As such the time rate of change of momentum is $\dfrac{mv-mu}{t}=\dfrac{m(v-u)}{t}$

Now, the second law gives

$$P \propto \text{time rate of change of momentum}$$

$$\Rightarrow P \propto \frac{m(v-u)}{t}$$

$$\Rightarrow P \propto \frac{m(u+\text{ft}-u)}{t} \qquad \text{since } v = u + \text{ft}.$$

$$\Rightarrow P \propto mf \qquad f \text{ being, the uniform acceleration.}$$

$$\Rightarrow P = kmf; \text{ where } k \text{ is the constant of proportionality.}$$

Choosing the units of mass, acceleration and force such that unit force produces unit acceleration on unit mass we get $1 = k.1.1$, giving $k = 1$.

Thus, $P = mf$. It is to be noted that Newton's second law holds even when m changes. But $P = mf$ does not hold in that case.

In the FPS system, the unit of force is one poundal and it is the force producing an acceleration of one foot per second per second on a mass of one pound when it acts continuously for one second.

In the CGS system, it is a dyne and it is the force which produces an acceleration of one centimeter per second per second on a mass of one gram when it acts continuously for one second.

In the MKS system, it is one Newton and it is the force producing an acceleration of one metre per second per second on a mass of one kilogram when it acts continuously for one second. It is denoted by 1N.

Newton's second law of motion is a quantitative law. It quantifies the measure of a force in terms of the mass and the acceleration that it produces on the body.

The equation $P = mf$ is sometimes called the kinetic equation in view of its fundamental importance in the study of kinetics.

The unit in which the force P in the kinetic equation $P = mf$ is quantified as the product of the mass and the acceleration produced by it on the mass is called the absolute or dynamical unit of force. One dynamical unit of force in the FPS system is 1 poundal, it is 1 dyne in the CGS system and 1 newton in the MKS system.

(Note that $1N = 1000 \times 100$ dynes $= 10^5$ dynes and 1 megadyne $= 10^6$ dynes $= 10$ N)

8.19 Weight of a body and the gravitational unit of force

The weight of a body is the resultant force with which the body is attracted towards the centre of the Earth.

Since, weight is a force, it must produce an acceleration on the body. In fact a body at a higher position from the surface of the earth when released falls towards the Earth due to the attraction of the Earth. While approaching towards the Earth the body does not maintain a uniform velocity. Rather it approached with an acceleration which is taken more or less uniform for considerable heights from the surface of the Earth.

This acceleration is called the acceleration due to gravity. It is denoted by g and its value is taken to be 32 ft/sec^2 hi the FPS system, 981 cm/sec^2 in the CGS system and 9.81 m/sec^2 in the MKS system.

A body of mass 1 1b on the surface of the Earth is attracted to the centre of the Earth by a force called 1 pound weight denoted by 1 Ib.wt. This force is equal to 1×32 poundals as given by $P = mf$.

Thus, 1 1b.wt. = 32 poundals, in absolute units.

Similarly, 1 gm.wt. is a force with which a body of mass 1 gm on the surface is attracted towards the centre of the Earth and accordingly,

1 gm wt = 981 dynes, in absolute units

Likewise, 1 kg.wt. is a force with which a body of mass 1 kg on the surface of the Earth is attracted to the centre of the Earth.

Surely, 1 kg. wt. = 9.81 Newtons in absolute units.

The unit of force expressed in terms of the gravitational pull on the mass is called the gravitational or statical unit of force. E.g. the gravitational pull on a mass of m lbs is m lb.wt.

One gravitational unit of force in the FPS system is 1 lb.wt, it is 1 gm.wt in CGS system and 1 kg.wt in MKS system.

You will come across soon that the weight of a body at different places of the Earth and at different altitudes are different This is due to the change of the acceleration due to gravity with the change of the distance of the place under consideration from the centre of the Earth.

At a place where the acceleration due to gravity is g the weight W of a body of mass m is given by, $W = mg$, in absolute units. It is to be noted that W changes with the change of g although m remains unchanged.

8.20 Newton's Third law of Motion

In view of the slight difference in the nature of the third law from the first two, we discuss the same separately.

Statement

To every action, there is an equal and opposite reaction.

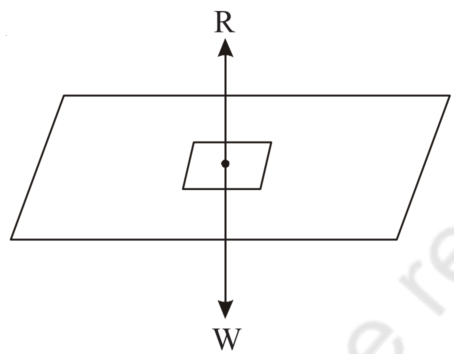
This law is both qualitative and quantitative in nature. The law essentially deals with two forces which are equal in magnitude but opposite in direction.

Many activities and phenomena in our daily life can be explained by this law.

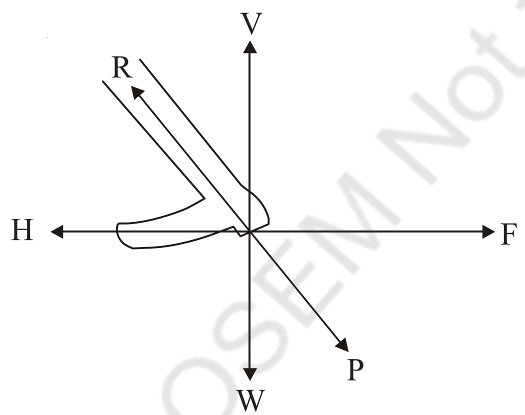
A few illustrations are given below :

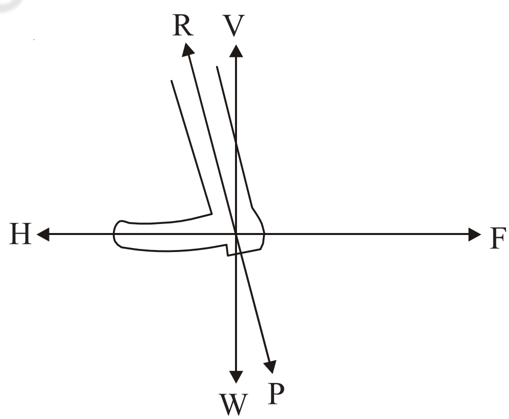
(a) When a body, say, a brick is on a table, the weight of the brick is giving a thrust on the table. Here, the thrust is the action. The third law says that the table is also giving a counter thrust on the brick in the opposite direction. This counter thrust is the reaction of the table on the brick.

If the table is in a horizontal position then, the weight, say W is acting vertically downwards. If R is the reaction of the table, then, W = R and R is vertically upwards.



(b) A man while walking on a horizontal road pushes on the ground in a slant direction through one of his legs.





If P is the force given by the man on the ground through one of his legs, then the reaction say R of the ground is just equal and oposite to P, so that P = R.

The vertical component say, V of R balances the weight say, W of the man. In fact, W is the vertical component of P. The horizontal component of P and the horizontal component say, H of R are equal and opposite. If F be the total frictional force then, the man can move if H > F. The net force H – F = *mf* where *m* is the mass of the man and *f* is the acceleration with he moves forward.

Notes :

i. It should be noted that action and reaction act on different bodies. In other words they cannot act on the same body.

ii. A frictional force changes its value keeping a maximum limit. Its direction also changes while opposing the acting force.

(c) When a man of mass m descends in a lift with an acceleration f, we see that the weight say, W, of the man is pressing downwards on the floor of the lift and lift is giving a reaction R on the man vertically upwards so that the net producing motion is W – R. Thus, from the Kinetic equation we get,

$$W - R = mf$$

$$\text{or,}\quad mg - R = mf$$

$$\Rightarrow \quad R = m(g - f)$$

If the lift falls freely, then $f = g$, giving R = 0. In this case there is no reaction of the lift on the man.

Again if $f = 0$ i.e. either there is no motion or the lift descends uniformly then $R = mg$, the weight of the man.

Again if the lift ascends with an acceleration f_1 then the reation R_1 of the lift is given by

$$R_1 - W = mf_1$$

$$\text{or,}\quad R_1 - mg = mf_1$$

$$\Rightarrow \quad R_1 = m(g - f_1)$$

When $f_1 = 0$, we get $R_1 = mg$ as in the earlier case.

Worked out examples

Example 1 : A body of mass 5 kg at rest is acted on by a constant force for 10 seconds and during this time it describes a distance of 150 m. Find the magnitude of the constant force and also the velocity of the body at the end of the time.

Solution : Let f m/sec^2 be the acceleration produced by the force say, P newtons.

Then using the formula, $s = ut + \frac{1}{2}ft^2$ we get,

$$150 = 0 + \frac{1}{2}f \times 10^2$$

Giving $f = 3$ m/sec^2

$\therefore$ the force P is given by

$$P = 5 \times 3$$
$$= 15 \text{ N}$$

If v m/sec be the velocity of the body at the end of the time i.e. 10 sec, then, from $v = u + ft$ we get,

$$v = 0 + 3 \times 10$$

i.e. $v = 30$ m/sec.

Example 2 : A body falls freely from a height h and comes to rest after penetrating into loose sand a depth d. Find in gravitational units the resistance per unit mass offered by the sand, supposed to be uniform.

Solution : Let m be the mass of the body.

As the result of the free fall through a height h, the velocity v acquired by the body just before penetrating into the sand is given by, $v^2 = 2\,gh$.

If R in absolute units be the total resistance offered by the sand on the body then, the effective force causing retardation on the body is $R - mg$, R being upwards and mg downwards.

If f be the retardation caused by the effective retarding force, then,

$$R - mg = mf$$

i.e. $$f = \frac{(R - mg)}{m}$$

Since the body comes to rest after penetrating through a depth d, we have

$$0^2 = v^2 - 2\,fd$$

i.e. $$2\,gh = 2\,fd$$

$$\Rightarrow gh = \frac{1}{m}(R - mg)d$$

$$\Rightarrow \frac{mgh}{d} = R - mg$$

$$\Rightarrow R = mg\left(1 + \frac{h}{d}\right) \text{ in absolute units.}$$

$$= m\left(1 + \frac{h}{d}\right) \text{ in gravitational units.}$$

$\therefore$ Resistance per unit mass $= \left(1 + \frac{h}{d}\right)$ in gravitational units.

EXERCISE 8.3

1. Find the acceleration produced by a constant force of 10 newtons on a mass of 5 kgs. Also find the velocity acquired by the body when the force acts for 10 seconds if initially the body was moving with a uniform velocity of 5 m/sec.

2. A heavy body of 80 kgs is being raised from the bottom of a pit 109 m deep by a uniform force of 120 kg. wt. Find the time taken by the body to reach the top of the pit.

3. Find the magnitude of the uniform force which increases the velocity of a body of mass 5 kgs from 20 m/sec to 30 m/sec in 10 minutes.

4. A mass of 4 kgs falls freely for 200 m and then brought to rest by penetrating 2 *m* in the mud. Find the average thrust of the mud on the body.

5. A truck of mass 5 (metric) tons and moving at a uniform rate of 60 m/ sec is brought rest in 10 seconds by the uniform application of brakes. Find the distance covered by the truck during this time and also calculate the magnitude of the force of resistance developed by the brakes in megadynes.

6. A body falls freely through a distance of 10 m from rest. It is then brought to rest in 1 second by a vertical force. A similar second force can bring the body to rest in 2 seconds. Show that the first force is $\frac{17}{12}$ times the second force.
   [Take g = 980 cm/sec^2]

7. A thin glass plate can just support a weight of 20 kgs. A body is placed on it and the plate is raised with the body on it with gradual increasing acceleration. It is observed that the plate just breaks when the accelaration is 6.54 m/sec^2. Find the mass of the body.

8. A body of mass *m* is put on another body of mass M and they are falling freely. Assuming that the air offers no resistance, show that one body does not give any pressure on the other.

ANSWERS

1. 2 m/sec^2, 25 m/sec.

2. $6\frac{2}{3}$ sec

3. $\frac{1}{12}$ N

4. 404 kg. wt

5. 300 m, 3000

7. 12 kg.